U.G. 3rd Semester Examination - 2021

MATHEMATICS

[PROGRAMME]

Course Code : MATH-G-CC-T-03

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

Notations/Symbols have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) Define "countable set" with an example.

b) If S be a non-empty subset of the Set $\boldsymbol{R}$ of real numbers which is bounded above with supremum$M$, then show the set

$$\{x \in \boldsymbol{R} : -x \in S\}$$

is bounded below and find its infimum.

c) Show that the set of all natural numbers is not bounded above.

d) Find all cluster points of the set $\left\{\frac{1}{n} : n \in \boldsymbol{N}\right\}$.

e) Show that the sequence $\{\sin nx\}$ is bounded, but not convergent.

f) Show that convergent sequence satisfies Cauchy convergence criterion for sequence.

g) Show that the sequence $\left\{\frac{n^2}{1+n^2}\right\}$ converges to 1.

h) If $\{x_n\}$ is a convergent sequence and c is a real number, then show that the sequence $\{cx_n\}$ is convergent.

i) Show that the sequence $\left\{\frac{n}{1+n}\right\}$ is a monotone sequence.

j) Find the sequence $\{s_n\}$ of partial sums of the series

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots$$

k) Show that the series

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots$$

is convergent with sum 2.

l) Find radius of convergence of the power series

$$1+x+x^2+x^3+\ldots$$

m) Show that the sequence $\{x^n\}$ is pointwise convergent on $(-1, 1]$.

n) Show that the sequence $\left\{\frac{x^n}{1+x^n}\right\}$, $x\in[0, 2]$ is not uniformly convergent on $[0, 2]$.

o) Show that the series $1+x+x^2+x^3+\ldots$, $x\in[0, 1]$ is pointwise convergent, but not uniformly convergent in [0, 1].

2. Answer any **four** questions: 5×4=20

a) State and prove the Archimedean property of ***R***.

b) If S and T be two non-empty bounded above subsets of ***R*** with respective supremum values p and q, then show that the set

$$S+T=\{s+t : s\in S, t\in T\}$$

is bounded above. Find the supremum of the set $S+T$.

c) State and prove the squeeze theorem of sequence of real numbers.

d) Show that the sequence

$$\left\{\left(1+\frac{1}{n}\right)^n\right\}$$

is convergent.

e) State and prove the Cauchy convergence criterion for series.

f) Show that the series

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots,\ p>0,$$

is convergent if $p>1$ and divergent if $0<p\le 1$.

g) State and prove Weierstrass' M-test for series of functions.

3. Answer any **two** questions: 10×2=20

a) i) Show that [0, 1] is not a countable set.

ii) State Bolzano-Weierstrass theorem for subset of real numbers. Give an example of a unbounded set with infinitely many cluster points. 6+(2+2)

b) i) State and prove the Cauchy convergence criterion for sequence.

ii) Prove that every convergent sequence has only one limit.

iii) Prove that every convergent sequence is bounded. 5+2+3

c) i) If a series of positive terms is convergent, then the sequence of its terms converges to zero.

ii) State Cauchy's root test of convergence of series. Prove that the series

$$1+\frac{1}{2^3}+\frac{1}{2^2}+\frac{1}{2^5}+\frac{1}{2^4}+\dots\dots\dots$$

is convergent.

iii) State Leibnitz's test of alternating series. 3+(2+3)+2

d) i) Show that the series

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n(n+1)}$$

is uniformly convergent for all real x.

ii) For the series

$$\sum_{n=1}^{\infty} f_n(x), \text{ where}$$

$$f_n(x)=n^2xe^{-n^2x^2}-(n-1)^2\,xe^{-(n-1)^2x^2},\quad x\in[0,\,1],$$

show that

$$\sum_{n=1}^{\infty}\int_0^1 f_n(x)\,dx \neq \int_0^1\left[\sum_{n=1}^{\infty} f_n(x)\right]dx$$ 4+6

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