U.G. 5th Semester Examination - 2022

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-11

(Partial Differential Equations & Applications)

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The notations and symbols have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) Obtain the partial differential equation from $z = f\left(x^2 - y^2\right)$ by eliminating the arbitrary function f.

b) Show that the function $u = \left(x^2 + y^2 + z^2\right)^{-\frac{1}{2}}$ satisfies the Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$.

c) Prove that the characteristic curves for the equation $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = u$ in xy plane are circles with centre at origin.

$-\frac{1}{2} - 1$ $-\frac{5}{2} + 1$

$\frac{-5+2}{2}$

$-\frac{3}{2} - 1$ $-\frac{5}{2}$

[Turn Over]

d) Define complete integral, general integral and singular integral of a first order partial differential equation $F(x, y, z, p, q)=0$.

e) Solve the equation

$$\left(x^2z-y^3\right)dx+3xy^2dy+x^3dz=0$$

first showing that it is integrable.

f) Find complete integral of the equation $pqz=p^2\left(xq+p^2\right)+q^2\left(yp+q^2\right)$.

g) If the operator $F(D, D')$ is reducible then show that the order in which the linear factors occur is unimportant.

h) Show that the equations $xp-yq=x$, $x^2p+q=xz$ are compatible.

i) Define a first order quasilinear partial differential equation with example.

j) When we say that a first order partial differential equation is separable?

k) Solve the partial differential equation $xu_x+yu_y+zu_z=xyz$.

$P\,dx + Q\,dy + R\,dz = 0$

l) Show that if f and g are arbitrary functions of a single variable, then

$$u = f(x - vt + iay) + g(x - vt - iay)$$

is a solution of the equation $u_{xx} + u_{yy} = \frac{1}{c^2} u_{tt}$

where $a^2 = 1 - \frac{v^2}{c^2}$.

m) What is meant by orthogonal trajectories on a surface?

n) Find the differential equation of all spheres of radius r having centre in the xy plane.

o) Define Cauchy problem for second order partial differential equation with example.

2. Answer any **four** questions: 5×4=20

a) Obtain the integral surface of the equation

$$(x - y)y^2 p + (y - x)x^2 q = (x^2 + y^2)z$$

through the curve $xz = a^3$, $y = 0$.

b) Apply Charpit's method to find the complete integral of $z^2 = pqxy$.

c) Solve $\frac{\partial U}{\partial x} = 2\frac{\partial U}{\partial t} + U$, $U(x, 0) = 6e^{-3x}$ by method of separation of variables.

d) Solve the equation

$$\frac{\partial^3 z}{\partial x^3}-2\frac{\partial^3 z}{\partial x^2\partial y}-\frac{\partial^3 z}{\partial x\partial y^2}+2\frac{\partial^3 z}{\partial y^3}=e^{x+y}.$$

e) Solve the one-dimensional diffusion equation

$\frac{\partial T}{\partial t}=\alpha\frac{\partial^2 T}{\partial x^2}$ in the region $0\le x\le \pi$, $t\ge 0$

subject to the conditions

i) T remains finite as $t\to\infty$

ii) $T=0$ if $x=0$ and π for all t

iii) At $t=0$, $T=\begin{cases} x, & 0\le x\le \pi/2 \\ \pi-x, & \pi/2\le x\le \pi. \end{cases}$

f) Show that a necessary and sufficient condition that the Pfaffian differential equation $X\cdot dr=0$ should be integrable is that $X\cdot Curl\, X=0$.

3. Answer any **two** questions: 10×2=20

a) i) Reduce the equation

$$y^2 z_{xx}-2xyz_{xy}+x^2 z_{yy}=\frac{y^2}{x}z_x+\frac{x^2}{y}z_y$$

to canonical form and hence solve it.

$2u\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$

ii) For what values of x and y the partial differential equation

$$z_{xx} + 2xz_{xy} + \left(1 - y^2\right) z_{yy} = 0$$

is hyperbolic, parabolic or elliptic?

8+2

b) i) Prove that the general solution of the linear partial differential equation $Pp + Qq = R$ is $F(u, v) = 0$ where F is an arbitrary function and $u(x, y, z) = c_1$, and $v(x, y, z) = c_2$ form a solution of the equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

ii) Find the general integral of the linear partial differential equation

$$(y + zx)p - (x + yz)q = x^2 - y^2 .$$ 6+4

c) i) Show that the general solution of the wave equation $u_{xx} = \frac{1}{c^2} u_{tt}$ is given by

$$u(x, t) = \frac{1}{2}\{f(x + ct) + f(x - ct)\} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\theta)\, d\theta$$

where initial deflection is $u(x, 0) = f(x)$ and initial velocity is $u_t(x, 0) = g(x)$.

ii) A string is stretched between two fixed points at a distance l apart. Motion is started by displacing the string in the form $y = a\sin(\pi x/l)$ from which it is released at time $t = 0$. Show that the displacement of any point at a fixed distance x from one end at time t is given by $y(x, t) = a\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi ct}{l}\right)$.

4+6

d) Find the solution of the equation $u_{xx} + u_{yy} = 0$ in a rectangle $0 \le x \le a$, $0 \le y \le b$ which satisfies the following boundary conditions $u(0, y) = u(a, y) = u(x, b) = 0$, $u(x, 0) = kx$ where k is a constant. 10
