U.G. 5th Semester Examination - 2022

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-12

(Group Theory-II)

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks. Symbols have their usual meanings.

GROUP–A

[Marks : 20]

1. Answer any **ten** questions: $2\times10=20$

a) If G be an abelian group and $f: G \to G$ be such that $f(x)=x^{-1}$ then show that f is an automorphism.

b) Find $Inn(S_3)$.

c) Define characteristic subgroup of a group.

d) Prove that center of a group G is a characteristic subgroup of G.

e) Prove that the commutator subgroup G' of a group G is a characteristic subgroup of G.

f) Define external direct product of groups.

g) Is the direct product $\mathbb{Z} \times \mathbb{Z}$ a cyclic group? Justify your answer.

h) Show that all abelian groups of order 22 are cyclic.

i) Let G be a group and S be a G-set. For all $a \in S$, prove that the subset $G_a = \{g \in G \mid ga = a\}$ is a subgroup of G.

j) If G be a finite group having only two conjugacy classes, then show that $|G| = 2$.

k) Whether 8 = 1 + 1 + 3 + 3 can be a class equation for a group? Justify your answer.

l) If G is a group of order 99, prove that G has a unique normal subgroup of order 11.

m) State Sylow's First Theorem.

n) What is Sylow p-subgroup of a group?

o) Show that center of a group of order 65 is isomorphic to $\mathbb{Z}_{65}$.

GROUP–B

[Marks : 20]

2. Answer any **four** questions: 5×4=20

   a) Find $Aut(\mathbb{Z}_3)$.

   b) Show that a characteristic subgroup of a group G is a normal subgroup of G. Is the converse true? Justify your answer. 3+2

   c) Let $G_1, G_2, ..., G_n$ be n groups. Prove that the group $G = G_1 \times G_2 \times ... \times G_n$ is abelian if and only if each of the groups $G_1, G_2, ..., G_n$ is abelian.

   d) Let S be a G-set, where G is a group and S is a nonempty set. Define a relation ρ on S by for all $a, b \in S$, $a\rho b$, if and only if $ga = b$ for some $g \in G$. Prove that ρ is an equivalence relation on S.

   e) Find all abelian groups of order 96.

   f) If G is a group of order pn, where p is prime such that $p \geq n$, then show that G has a normal subgroup of order p.

GROUP–C

[Marks : 20]

3. Answer any **two** questions: $10 \times 2 = 20$

a) i) Let G be a group and Z be the centre of the group G. Prove that set of all innerautomorphisms of G is isomorphic to the quotient group G/Z. 5

ii) If $|Aut(G)| > 1$, then show that $|G| > 2$. 3

iii) Show that $Aut(\mathbb{Z}_5) \cong \mathbb{Z}_4$. 2

b) i) If G is the internal direct product of the normal subgroups $N_1, N_2, \ldots, N_k$, then show that $G \cong N_1 \times N_2 \times \ldots \times N_k$. 5

ii) Find the number of elements of order 5 in $\mathbb{Z}_{15} \times \mathbb{Z}_5$. 5

c) i) If G be a group of order pq, where p, q are primes such that $p > q$ and q does not divide $p - 1$, then show that G is a cyclic group. 5

ii) Show that no group of order 56 is a simple group. 5

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