U.G. 2nd Semester Examination - 2019

MATHEMATICS

[HONOURS]

Course Code : MTMH/CC-T-03

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The symbols and notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) If A and B are bounded subsets of $\mathbb{R}$, then prove that $A \cap B$ and $A \cup B$ are also bounded.

b) Prove that the set $\{\pm 1, \pm 4, \pm 9, \pm 16, \ldots\}$ is countable.

c) State completeness axiom of $\mathbb{R}$. Let S be a non-empty, bounded above subset of $\mathbb{R}$. Show that the set $T = \{-x : x \in S\}$ is bounded below.

d) Give an example of an open set which is not an interval.

e) Construct a set $S \subseteq \mathbb{R}$ such that $S' = \mathbb{N}$, where S' is the derived set of S.

[Turn Over]

f) Give an example of an open cover of the set $[0, \infty)$ which does not have a finite sub-cover.

g) State Archimedean property of real numbers and hence show that $\frac{1}{n} \to 0$ as $n \to \infty$.

h) Given that $y_n \to 0$ as $n \to \infty$. If $\lim_{x \to \infty} \frac{x_n}{y_n}$ is finite, then show that $x_n \to 0$ as $n \to \infty$. Show by an example that the converse is not true.

i) Show that the sequence $\left\{\left(1-\frac{1}{n}\right)\text{Cos}\frac{n\pi}{2}\right\}$ is not convergent but has a convergent subsequence.

j) Let $x_n = 1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}, \forall\, n \in \mathbb{N}$. Prove or disprove $\{x_n\}$ is a cauchy sequence.

k) A sequence $\{x_n\}$ is defined as follows:

$$x_1 \leq x_3 \leq x_5 \leq \ldots \leq x_6 \leq x_4 \leq x_2.$$

If $(x_{2n} - x_{2n-1}) \to 0$ as $n \to \infty$, show that $\{x_n\}$ is convergent.

l) Give examples of:

i) a convergent series $\sum_{n=1}^{\infty} a_n$ of positive terms such that $\lim_{n\to\infty} \frac{a_n+1}{a_n} = 1$;

ii) a divergent series $\sum_{n=1}^{\infty} b_n$ of positive terms such that $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = 1$.

m) If $\sum_{n=1}^{\infty} u_n$ is a convergent series of positive terms, show that $\sum_{n=1}^{\infty} \frac{u_n}{1+u_n}$ is convergent.

n) If $\sum_{n=1}^{\infty} u_n$ be absolutely convergent and $\{v_n\}$ be a bounded sequence, then show that the series $\sum_{n=1}^{\infty} u_n v_n$ is absolutely convergent.

o) Prove that the series $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots$ is conditionally convergent.

2. Answer any **four** questions: $5\times4=20$

a) i) Let S be a non-empty bounded subset of $\mathbb{R}$ with sup S = M and inf S = m. Prove that the set $T=\{|x-y| : x, y \in S\}$ is bounded above and sup T = M−m.

ii) Give an example of an infinite set $S \subseteq \mathbb{R}$ such that S has only one limit point.

(1+3)+1

b) i) Let A be a non-empty subset of $\mathbb{R}$ and $d(x, A)= \inf\{|x-y| : y \in A\}$. Prove that $d(x, A) = 0$ if and only if $x \in \bar{A}$.

ii) Give an example of a family $\{I_n : n \in \mathbb{N}\}$ of non-empty closed intervals such that

$$I_1 \supset I_2 \supset I_3 \supset \ldots \text{ and } \bigcap_{n=1}^{\infty} I_n = \phi.$$ 4+1

c) i) If $\{x_n\}$ be a sequence such that $\lim\limits_{x\to\infty} \frac{x_n+1}{x_n} = l$ where $|l|<1$, then show that $\lim\limits_{n\to\infty} x_n = 0$.

ii) Show that $\lim\limits_{n\to\infty} \frac{x^n}{n} = 0$, if $|x| \leq 1$.

4+1

d) Prove that the sequence $\{x_n\}$ defined by $x_1 = \sqrt{7}$ and $x_{n+1} = \sqrt{7 + x_n}$, $\forall n \geq 1$ converges to the positive root of the equation $x^2 - x - 7 = 0$. 5

e) If Σx_n be a convergent series of positive real numbers and $\{x_n\}$ is a monotonic decreasing resonance, then show that $\lim_{n \to \infty} n x_n = 0$. 5

f) i) If the series Σx_n is convergent, then show that $\lim_{x \to \infty} |u_n| = 0$.

ii) If $\alpha > 0$, $\beta > 0$, test the convergence of the series

$$1 + \frac{\alpha+1}{\beta+1} + \frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)} + \frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)} + \ldots$$

2+3

3. Answer any **two** questions: 10×2=20

a) i) Show that the intersection of a finite number of open sets in $\mathbb{R}$ is an open set. Give an example to show that the intersection of an infinite number of open sets in $\mathbb{R}$ is not necessarily an open set.

ii) Let K be a compact subset of $\mathbb{R}$ and $F \subset K$ be a closed subset in $\mathbb{R}$. Prove that F is compact in $\mathbb{R}$. (4+2)+4

b) i) If a sequence $\{x_n\}$ is monotone increasing and bounded above, then show that it is convergent and converges to its least upper bound.

ii) Prove that in $\mathbb{R}$, every cauchy sequence is convergent. Is it true in the set of rational number Q? Justify your answer.

iii) Prove that $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)=0$.

3+(4+1)+2

c) i) Let Σx_n be a series of positive real numbers and $\lim_{n\to\infty} x_n^{\frac{1}{n}} = l$. Then show that Σx_n is convergent if $l < 1$ and Σx_n is divergent of $l > 1$.

ii) Examine the convergence of the series $x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots, x>0$.

iii) Prove that an absolutely convergent series is convergent. 5+3+2

d) i) Show that the unit interval [0, 1] is uncountable.

ii) Find $\lim_{n\to\infty} \inf x_n$ and $\lim_{n\to\infty} \sup x_n$ where

$$x_n = \frac{1}{n} + \sin\frac{n\pi}{4}.$$

iii) If $0 \le a < b < 1$, then show that the infinite series $1 + a + ab + a^2b + a^2b^2 + \ldots$ is convergent. 5+3+2
