U.G. 5th Semester Examination - 2020

MATHEMATICS

[PROGRAMME]

Discipline Specific Elective (DSE)

Course Code : MATH-G-DSE-T-1B

(Complex Analysis)

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks. Candidates are required to give their answers in their own words as far as practicable.

Unless otherwise stated notations carry their usual meanings.

1. Answer any **ten** questions from the following: 2×10=20

a) Find $\lim\limits_{z\to -3+i\sqrt{2}} \dfrac{z+3-i\sqrt{2}}{z^2+6z+11}$.

b) Examine the continuity of the function f defined by $f(z) = \frac{Re(z)}{z+iz} - 2z^2$ at $z_0 = e^{i\pi/4}$.

c) Give an example to show that the continuous function may not be uniformly continuous.

d) Show that the function $f(z) = x^2 + y^2$ is not analytic at any point.

e) If *f(z)* and $\overline{f(z)}$ are both analytic in a domain *D* then prove that *f(z)* is constant in *D*.

f) Find the radius of convergence of the power series $\sum\limits_{n=0}^{\infty} a_n z^n$, where

$$\begin{aligned} a_n &= i2^n \quad \text{for even } n \\ &= -3^n \quad \text{for odd } n. \end{aligned}$$

g) Prove that $\left|\int_C \frac{dz}{z^2+10}\right| \leq \frac{2\pi}{3}$, where C is the circle $C : z(t) = 2e^{it}, (-\pi \leq t \leq \pi)$.

h) Does there exist a function *f(z)* analytic in $|z| < 1$ and satisfying $f(\frac{1}{2n}) = f(\frac{1}{2n+1}) = \frac{1}{2n} (n = 1, 2, ...)$?

i) Evaluate $\int_C (3z^2 - 2z)dz$, where C is the contour defined by $z(t) = t + it^2, t \in [0, 1]$.

j) Suppose f is analytic for $|z| \leq 1,\ f(0) = 0$ and $f(z) \leq 5$ for all $|z| = 1$. Can $|f'(0)| > 5$?

k) If *P(z)* is a polynomial of degree *n*, prove that $\int_{|z|=2} \frac{P(z)}{(z-1)^{n+2}} dz = 0$.

l) Suppose $f(z)$ and $g(z)$ are entire functions, $g(z)$ is never zero and $|f(z)| \leq |g(z)|$ for all *z*. Show that there exist a constant *c* such that $f(z) = cg(z)$.

m) Show that the range of a non-constant entire function is dense in $\mathbb{C}$.

n) At which points is the function $f(z) = zRe(z)$ differentiable?

o) Let f be analytic in a domain D, then all the derivatives of f exist and are analytic in D. Justify the statement.

2. Answer **four** questions from the following: 5×4=20

a) Let $f = u + iv$ be differentiable in an open connected set D and assume that the real and imaginary part of f are related by $au + bv + c = 0$, where the numbers a, b, c are real, a and b not being simultaneously zero. Find f.

b) State and prove Liouville's theorem.

c) Prove that the zeros of a non-constant analytic function are isolated points.

d) Find the Laurent expansion of the function $\frac{e^z}{z(z^2+1)}$ in the domain $0 < |z| < 1$.

e) Prove that the sequence $\{f_n\}$, where $f_n(z) = \frac{1}{1+nz}$ converges uniformly in the region $|z| \geq 2$ but not uniformly in the region $|z| \leq 2$.

f) State Cauchy's Integral formula. Use it to evaluate $\int_C \frac{z^3+3z-1}{(z-1)(z-2)} dz$, where C is the circle $|z| = 3$.

3. Answer **two** questions from the following: 10×2=20

a) i) Prove that if a function f is continuous in a closed and bounded region D then f is bounded in D. Is it true if the domain is not closed? 4+1

ii) Discuss the continuity of the function

$$f(z) = \frac{(Re(z))^2(Im(z))}{|z|^2} \quad \text{if } z \neq 0$$
$$= 0 \quad \text{if } z = 0$$

at the all points of $\mathbb{C}$. 5

b) i) Show that at $z = 0$ the function defined by

$$f(x+iy) = \frac{(1+i)x^3-(1-i)y^3}{x^2+y^2} \quad \text{for } x+iy \neq 0$$
$$= 0 \quad \text{for } x+iy = 0$$

satisfies the Cauchy-Riemann equations but it is not differentiable. 5

ii) Discuss the convergence of the series

$$\sum_{n=-\infty}^{\infty} \frac{z^{2n}}{3^{|n|}}.$$ 5

c) i) If $f(z)$ be a nonzero analytic function in a simply connected domain D then prove that there exists a function $g(z)$, analytic in D, such that $e^{g(z)} = f(z)$. 3

ii) Evaluate $\int_{|z|=3} \frac{3z^4+2z-6}{(z-2)^3} dz$. 3

iii) Prove that a sequence of functions convergence uniformly on a set E if and only if the sequence is uniformly Cauchy in E. 4

d) i) Let
$$f(z) = \frac{x^3y(y-ix)}{x^6+y^2}, \quad \text{if } z \neq 0,$$
$$= 0, \quad \text{if } z = 0.$$

Show that $\lim_{z\to 0} \frac{f(z)-f(0)}{z}$ exists along any fixed direction, that all these limits are equal to zero, but that f is not differentiable at the origin. Is it continuous there? 2+2+2

ii) Define analytic function at point. If $f(z)$ is analytic in a domain D and $Re(f(z))$ is constant in D then prove that $f(z)$ is constant in D.

1+3

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