

U.G. 1st Semester Examination - 2022

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-02

(Algebra)

Full Marks : 60

Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The notations and symbols have their usual meanings.

GROUP-A

[Marks : 20]

1. Answer any ten questions: 2×10=20

a) Find the principal argument of $z = 1 + i \tan \frac{3\pi}{4}$.

~~b)~~ Find the cube roots of -1 .

~~c)~~ Find the general value of i^n .

d) Show that $\sin \left\{ i \log \left(\frac{a-ib}{a+ib} \right) \right\} = \frac{2ab}{a^2+b^2}$.

e) Find the equation whose roots are the cube of the roots of the equation $x^3 + 3x^2 + 2 = 0$.

$5 \times \frac{1}{2} = 5$

d) If $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^2 + 3$, $x \in \mathbb{R}$, then find $f^{-1}(3)$ and $f^{-1}(7)$. $f^{-1}(x) = y$

g) Give an example of a function which is neither one-one nor onto.

b) Find the order of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 1 & 3 & 4 \end{pmatrix}$$

i) If $a|bc$ and $\gcd(a, b) = 1$, then prove that $a|c$

j) Find the integers s, t such that

$$\gcd(33, 57) = 33s + 57t.$$

k) What is the remainder when $1! + 2! + 3! + \dots + 100!$ is divided by 15?

l) Find λ such that $A = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & \lambda \end{pmatrix}$ is an

orthogonal matrix.

m) Find all real values of λ for which the rank of

the matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & \lambda \\ 5 & 7 & 1 & \lambda^2 \end{pmatrix}$ is 2.

$$x = y^2 + 3$$

$$y = \sqrt{x-3}$$

$$f^{-1}(x) = \sqrt{x-3}$$

$$f^{-1}(3) = 0$$

$$f^{-1}(7) = \sqrt{4} = 2$$

$$f^{-1}(x) = \sqrt{x-3}$$

$$\begin{array}{l} a|ac \\ a|bc \\ a|acu+bcv \\ a|c \end{array}$$

$$\begin{array}{l} au + bv \perp \\ acu + bcv \perp \end{array}$$

$$\begin{array}{l} a|ac \\ a|bc \\ a|acu+bcv \\ a|c \end{array}$$

- ✓ n) In a group (G, \circ) , show that $a \circ b = a \circ c \Rightarrow b = c$ for all $a, b, c \in G$.
- o) In any group G if $a^3 = e$ and $aba^{-1} = b^2$ for $a, b \in G$, then find the order of b .

GROUP-B

[Marks : 20]

$\sqrt{1+4H^3}$

2. Answer any four questions: 5×4=20

a) State De Moivre's theorem. If

$$z_r = \cos \frac{\pi}{2^r} + i \sin \frac{\pi}{2^r} \quad (r = 1, 2, 3, \dots),$$

then prove that $z_1 \cdot z_2 \cdot z_3 \dots \infty = -1$. 2+3

b) Show that $n^2 < 2^n \quad \forall n \geq 5$. 5

✓ c) Solve by Cardan's method $x^3 - 12x + 65 = 0$. 5

✓ d) Define equivalence relation on a set S . Show that the relation

$$\rho = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a - b \text{ is divisible by } 5\}$$

is an equivalence relation on the set \mathbb{Z} of all integers. 2+3

✓ e) Define a group. Give an example of finite and infinite group. Does the set \mathbb{Z} of all integers

form a group under the operation defined as $a \circ b = a - b$ for all $a, b \in \mathbb{Z}$? Justify your answer. 2+1+2

Reduce the matrix $A = \begin{pmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$ to a row reduced echelon form and hence find its rank. 4+1

GROUP-C

[Marks : 20]

3. Answer any two questions: 10×2=20

a) i) Show that the sum of 99th powers of the roots of the equation $x^5 - 1 = 0$ is zero.

5

ii) Apply Descartes's rule of signs to show that the equation $x^7 + 5x^4 - 3x + 1 = 0$ has at least four imaginary roots.

3

iii) If one of the roots of the equation $x^3 + px^2 + qx + r = 0$ equals the sum of the other two, then prove that $p^3 + 8r = 4pq$.

2

b) i) Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two mappings such that $g \circ f$ is injective and f is surjective. Prove that g is injective.

3

ii) In a group (G, \circ) , prove that $(a^{-1})^{-1} = a$ and $(a \circ b)^{-1} = b^{-1} \circ a^{-1}$ for all $a, b \in G$. 3

iii) Show that the residue classes $[1], [3], [5], [7]$ modulo 8 form a multiplicative group.

4

c) i) Investigate for what values of λ and μ the following equations

$$x + y + z = 6$$

$$x + 2y + 3z = 10$$

$$x + 2y + \lambda z = \mu$$

$$\frac{-1 + \sqrt{31}}{2}$$

have no solution, a unique solution and an infinite number of solutions. 5

ii) Obtain the normal form under congruence and find the rank, signature of the

symmetric matrix $\begin{pmatrix} 2 & 4 & 3 \\ 4 & 6 & 3 \\ 3 & 3 & 1 \end{pmatrix}$. 5

$$\frac{1}{2} (\alpha \pm \sqrt{1 + 4\sqrt{3}})$$