U.G. 6th Semester Examination - 2022

MATHEMATICS

[HONOURS]

Discipline Specific Elective (DSE)

Course Code : MATH-H-DSE-T-03B

(Number Theory)

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The symbols and notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) Show that the cube of any integer is of the form $7k$ or $7k \pm 1$.

b) Prove that no integer in the following sequence is a perfect square:

$11, 111, 1111, 11111, \ldots$

c) Prove that if a and b are both odd integer then $16 | a^4 + b^4 - 2$.

d) Show that for a positive integer n and any integer a, gcd $(a, a+n)$ divides n.

e) Assuming that $\gcd(a,b)=1$, prove $\gcd(a+b, a^2+b^2) = 1$ or 2.

f) Divide 100 into two summands such that one is divisible by 7 and other by 11.

g) Find all prime numbers that divide 50!.

h) Prove that the only prime of the form $n^3 - 1$ is 7.

i) Prove that if n > 2, then there exists a prime p satisfying $n < p < n!$.

j) What is the remainder when

$1^5 + 2^5 + 3^5 + \cdots + 99^5 + 100^5$ is divided by 4?

k) Prove that $111^{333} + 333^{111}$ is divisible by 7.

l) Find all solutions of the linear congruence $3x - 7y \equiv 11 (mod\ 13)$.

m) Establish the congrunce

$2222^{5555} + 5555^{2222} \equiv 0 (mod\ 7)$.

n) If p is a prime and $k > 1$, show that $\varphi\left(\varphi(p^k)\right) = p^{k-2}\varphi((p-1)^2)$.

o) Show that if n is product of twin primes, $n = p(p+2)$, then $\varphi(n)\sigma(n) = (n+1)(n-3)$.

2. Answer any **four** questions: 5×4=20

a) Find integers x, y, z satisfying
$\gcd(198, 288, 512) = 198x + 288y + 512z$.

b) If $n > 1$ is an integer not of the form $6k+3$, prove that $n^2 + 2^n$ is composite.

c) For $n > 3$, show that the integers n, $n+2$, $n+4$ can not all be prime.

d) Establish that the sequence $(n+1)! - 2, (n+1)! - 3, \ldots, (n+1)! - (n+1)$, produces n consecutive composite integer for $n > 2$.

e) Find all values of $n \geq 1$ for which $n! + (n+1)! + (n+2)!$ is perfect square.

f) Assuming that 495 divides $273x49y5$, obtain the digits x and y.

3. Answer any **two** questions: 10×2=20

a) i) If a is an arbitrary integer then show that $6|a(a^2+11)$ and $360|a^2(a^2-1)(a^2-4)$. 4

ii) Establish that if a is an odd integer then for $n \geq 1, a^{2^n} \equiv 1 (mod\ 2^{n+2})$. 6

b) i) Find the value of $n \geq 1$ for which $1! + 2! + \cdots + n!$ is a perfect square. 4

ii) If p and q are distinct primes, prove that $p^{q-1} + q^{p-1} \equiv 1 (mod\ pq)$. 6

c) i) Establish that for any positive integer $n, \frac{\sigma(n!)}{n} \geq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. 5

ii) For any two positive integers m and n, where $d = \gcd(m, n)$, prove that $\varphi(m)\varphi(n) = \varphi(mn)\frac{\varphi(d)}{d}$. 5

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