U.G. 5th Semester Examination - 2021

MATHEMATICS

[PROGRAMME]

Skill Enhancement Course (SEC)

Course Code : MATH-G-SEC-T-3A&B

Full Marks : 40 Time : 2 Hours

The figures in the right-hand margin indicate marks. Candidates are required to give their answers in their own words as far as practicable.

Answer all the questions from selected Option.

OPTION-A

MATH-G-SEC-T-3A

(Integral Calculus)

1. Answer any **five** questions: 2×5=10

   a) If $f(x) \geq g(x)$ are integrable functions for all real $x \in [a\ b]$, then prove that $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

   b) Prove that $\int_0^2 |1-x|\,dx = 1$.

   c) If $f(x) = \int_0^x \cos^4 t\,dt$, prove that $f(x+\pi) = f(x) + f(\pi)$.

   d) Obtain a reduction formula for $\int \cot^n x\,dx$.

   e) Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\,dxdydz$.

   f) Find the area of the region enclosed by the curve $\sqrt{x}+\sqrt{y}=\sqrt{5}$ and the coordinate axes.

   g) Find the volume of the solid obtained by revolving the cycloid $x = a(\theta + \sin\theta)$ about its base.

   h) Find the perimeter of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

2. Answer any **two** questions: 5×2=10

   a) If $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$ (n being an integer greater than 1), then prove that $I_n = \frac{n-1}{n} I_{n-2}$.

   b) If $\phi(x) = \int_{-1}^{1} \frac{\sin x}{1+t^2}\,dt$, find $\phi'\left(\frac{\pi}{3}\right)$.

   c) Prove that the length of the arc of the curve $r = ae^{\theta\cot\alpha}$ between the radii vectors r_1 and r_2 is $(r_2 - r_1)\sec\alpha$.

   d) The area enclosed by $y^2 = 4x$ and $x^2 = 4y$ is revolved about the x-axis. If V be the volume of revolution, prove that $5V = 96\pi$.

3. Answer any **two** questions: 10×2=20

   a) Evaluate :

      i) $\int \frac{1+x^2}{(1-x^2)\sqrt{1+x^2+x^4}}\,dx$

      ii) $\int \tan^3 3x\,dx$. 5+5

   b) i) Evaluate $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}}\left\{1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}\right\}$. 5

ii) Prove that the area between the curve $y^2(a+x) = (a-x)^3$ and its asymptote is $3a^2\pi$. 5

c) i) Prove that the length s of an arc of the curve $x\sin\theta + y\cos\theta = f'(\theta)$, $x\cos\theta - y\sin\theta = f''(\theta)$ is given by $s = f(\theta) + f''(\theta) + c$, where c is a constant. 5

ii) Show that the volume of the solid formed by revolving the ellipse $x = a\cos\theta,\ y = b\sin\theta$ about the line $x = 2a$ is $4\pi^2a^2b$. 5

d) i) If s be the length of the curve $r = a\tanh\frac{\theta}{2}$ between the origin and $\theta = 2\pi$, and Δ be the area between the same points, show that $\Delta = a(s - a\pi)$. 5

ii) If $I_{m,n} = \int_0^1 x^m(1-x)^n\ dx$, where m and n are positive integers, prove that $(m+n+1)I_{m,n} = nI_{m,n-1}$. 5

OPTION-B
MATH-G-SEC-T-3B
(Vector Calculus)

1. Answer any **five** questions: $2\times5=10$

a) If $\vec{r} = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\hat{k}$ where $\hat{i},\hat{j},\hat{k}$ have their usual meaning, then find

$$\left(\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right).$$

b) If $\hat{a}$ is a unit vector in the direction of $\vec{b}$, a vector function of the scalar variable t, show that

$$\hat{a}\times\frac{d\hat{a}}{dt} = \left(\vec{b}\times\frac{d\vec{b}}{dt}\right)/(\vec{b}.\vec{b}).$$

c) Show that $\vec{F} = (2xy+z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force field.

d) If $\vec{F} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, find $\int_C \vec{F}.d\vec{r}$, from (0, 0, 0) to (1, 1, 1) along the path C given by $x = t, y = t^2, z = t^3$.

e) Show that if the vectors $\vec{\alpha},\vec{\beta}$ are irrotational, then the vector $\vec{\alpha}\times\vec{\beta}$ is solenoidal.

f) If $\vec{a}$ is constant vector, then prove that $curl(\vec{a}.\vec{r})\vec{a} = \vec{0}$.

g) If $\vec{A}$ has constant magnitude then show that $\vec{A}\times\frac{d\vec{A}}{dt}=0$.

h) Find a unit normal to the surface $2x^2y+3yz=4$ at the point (1, –1, –2).

2. Answer any **two** questions: 5×2=10

a) If $\frac{d\vec{a}}{dt}=\vec{r}\times\vec{a}$ and $\frac{d\vec{b}}{dt}=\vec{r}\times\vec{b}$ then show that $\frac{d}{dt}(\vec{a}\times\vec{b})=\vec{r}\times(\vec{a}\times\vec{b})$ where $\vec{r}$ is a constant vector and $\vec{a},\vec{b}$ are vector functions of a scalar variable *t*.

b) Find divergence and curl of the vector $\vec{v}=\frac{\hat{r}}{r}$, where $\vec{r}$ is the unit vector along $\vec{r}$ and *r* is the magnitude of the vector $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$.

c) Find the constants *p* and *q* such that the surfaces $px^2-qyz=(p+2)x$ and $4x^2y+z^3=4$ are orthogonal at (1, –1, 2) .

d) Find the equation of the tangent plane and normal line to the surface $x^2+y^2-z=0$ at the point $(2,-1,5)$.

3. Answer any **two** questions: 10×2=20

a) i) Determine *a, b* and *c* so that $\vec{u}$ is irrotational where

$$\vec{u}=(x+2y+az)\hat{i}+(bx-3y-z)\hat{j}+(4x+cy-2z)\hat{k}\cdot$$

ii) Let $\phi=x^3+y^3-z^3$ be a scalar point function. Verify that $curl\,(grad\,\phi)=\vec{0}$.

5+5

b) Evaluate $\iint_S \vec{A}\bullet\vec{n}dS$ where $\vec{A}=yz\hat{i}+zx\hat{j}+xy\hat{k}$ and *S* is the part of the surface $x^2+y^2+z^2=1$ which lies in the first octant.

c) i) Let *V* be the closed region bounded by the surfaces $x=0, x=2, y=0, y=6, z=x^2, z=4$ and $\vec{F}=y\hat{i}+2x\hat{j}-z\hat{k}$. Find $\iiint_V \vec{\nabla}\times\vec{F}dV$.

ii) If $\vec{\alpha}=(\sin\theta,\cos\theta,\theta), \vec{\beta}=(\cos\theta,-\sin\theta,-3)$ and $\vec{\gamma}=(2,3,1)$, find the value of $\frac{d}{d\theta}\left\{\vec{\alpha}\times\left(\vec{\beta}\times\vec{\gamma}\right)\right\}$ at $\theta=0$. 6+4

d) Prove that $\oint_C \vec{F}\bullet d\vec{r}=0$ for every closed curve *C* if and only if $\vec{\nabla}\times\vec{F}=\vec{0}$ if everywhere. 5+5

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