U.G. 3rd Semester Examination - 2020

MATHEMATICS

[HONOURS]

Generic Elective Course (GE)

Course Code : MATH-GE-T-01

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.
Symbols and Notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) Give an example of removable discontinuity.

b) Using $\varepsilon - \delta$ definition show that $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

c) If *f* is continuous at *c* and then show that $|f|$ is also continuous at *c*.

d) Prove that the equation $\frac{1}{x-1} + \frac{2}{x-2} + \frac{3}{x-3} = 0$ has one solution between 1 and 2 and another between 2 and 3.

e) Suppose *f(x)* is such a quadratic expression that it is positive for all real x . If $g(x) = f(x) + f'(x) + f''(x)$, then show that for all real x, $g(x) > 0$.

f) Find the angles of intersection between the curves $y = x^2$ and $y = 2 - x^2$.

g) Interpret Rolle's theorem geometrically.

h) Find the value of $\lim_{x\to 0} \frac{x^3}{\cosh x - 1}$.

i) Examine whether the L'Hospital rule is applicable on $\lim_{x\to 0} \frac{e^x - 1}{x}$.

j) Sketch the curve $r = ae^{\theta \cot\alpha}$.

k) Prove that the locus of the extremity of the polar subtangent of the curve $\frac{1}{r} + f(\theta) = 0$ is $\frac{1}{r} = f'\left(\frac{\pi}{2} + \theta\right)$.

l) Show that $x = -3a$ is a point of inflection of the curve $y^3 + 3ax^2 + x^3 = 0$.

m) Find equation of the asymptotes of the curve $x^2y^2 - 2x + 4 = 0$.

n) Show that the semi-vertical angle of the cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

o) If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

2. Answer any **four** questions: 5×4=20

a) Find the curvature at any point on the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. 5

b) If $f(x+y) = f(x) + f(y)$ for all x and y and $f(x)$ is continuous at $x=0$ Then show that $f(x)$ is continuous for all values of x. 5

c) Show that the eight points intersection of the curve $x^4 - 5x^2y^2 + 4y^4 + x^2 - y^2 + x + y + 1 = 0$ and its asymptotes lie on a rectangular hyperbola. 5

d) If $\phi(x)$ be a polynomial in x and λ is a real number then prove that there exists a root of $\phi'(x) + \lambda\phi(x) = 0$ between any pair of roots of $\phi(x) = 0$. 5

e) Consider the function $f(x, y)$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & where \quad x^2 + y^2 \neq 0 \\ 0, & where \quad x^2 + y^2 = 0 \end{cases}$$

Show that f is not differentiable at (0,0) though $f(x, y)$ is continuous there. 5

f) If $y = x^{n-1} e^{\frac{1}{x}}$ then prove that $D^n y = \frac{(-1)^n e^{\frac{1}{x}}}{x^{n+1}}$. 5

g) Find the pedal equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with respect to the centre as pole.

3. Answer any **two** questions: 10×2=20

a) i) If $x + y = 1$, then prove that n[th] derivative of $x^n y^n$ is

$$n!\left\{y^n - \binom{n}{1}^2 y^{n-1}x + \binom{n}{2}^2 y^{n-2}x^2 - \binom{n}{3}^2 y^{n-3}x^3 + \cdots + (-1)^n x^n\right\}$$ 5

ii) If $F=(x, y)=0$, then show that

$$\frac{d^2y}{dx^2} = -\frac{(F_y)^2 F_{xx} - 2F_x F_y F_{xy} + (F_x)^2 F_{yy}}{(F_y)^3}.$$ 5

b) i) If the perimeter of a triangle remains constant. Prove that the area of the triangle is greatest when the triangle is equilateral. 4

ii) If $z = f(u, v)$ where $u = x^2 - 2xy - y^2$ and $v = y$ show that $(x+y)\frac{\partial z}{\partial x} + (x-y)\frac{\partial z}{\partial y} = 0$ can be transformed into $\frac{\partial z}{\partial v} = 0$. 6

c) i) State and prove the Cauchy's Mean-Value theorem. 5

ii) Find the expansion of $(1+x)^n$ in a power series of x and indicate the range of validity of the expansion. 5

d) i) Determine

$$\lim_{x \to \infty} [x - \sqrt[n]{(x - a_1)(x - a_2) \ldots \ldots \ldots (x - a_n)}].$$ 5

ii) Show that at $x = \frac{1}{4}$ the function $f(x) = \frac{1}{8} log\, x - bx + x^2, x > 0$, where $b \geq 0$ is a constant has neither maximum nor minimum 5
