U.G. 1st Semester Examination - 2021

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-02

(Algebra)

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The notations and symbols have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) Find the sum of all 1729th roots of unity.

b) Expand $\cos^7\theta$ in a series of cosines of muiltiples of θ.

c) If z is a complex number such that $|z|=2$ then show that the point $z+\frac{1}{z}$ lies on an ellipse of eccentricity $\frac{4}{5}$ in the complex plane.

d) Prove that $7^{2n}-48n-1$ is divisible by 2304 for every natural number n.

e) If α, β, γ are roots of the equation $x^3+qx+r=0$ then find the equation whose roots are $\alpha(\beta+\gamma)$, $\beta(\gamma+\alpha)$, $\gamma(\alpha+\beta)$.

f) Discuss the nature of roots of the equation $x^4+4x^3-12x^2-32x+k$ for all values of k.

g) Prove that every prime greater than 3 can be written in the form $6n+1$ or $6n+5$.

h) Find last two digits of 7^{100}.

i) Find the sum of the absolute values of the roots of

$$x^{172920152016}=1.$$

j) Find the number of elements in the set $\{\sigma \in S_4 : \sigma(3)=3\}$.

k) Suppose that f is a mapping from a set S to itself and $f(f(x))=x$ for all x in S. Prove that f is one-to-one and onto.

l) Show that the group $GL(2, R)$ is non-abelian.

m) Find the inverse of 13 in $U(14)$.

n) If a vector in $\mathbb{R}^3$ is multiplied by a 3×3 orthogonal matrix, what happens to its magnitude and direction?

o) Find the real constants a and b in the matrix

$$Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1/\sqrt{2} & a \\ 0 & 1/\sqrt{2} & b \end{bmatrix}$$

such that Q is orthogonal.

2. Answer any **four** questions: 5×4=20

a) i) If $V_n = a^n + b^n$, where a and b are the roots of $x^2 + x + 1$, what is the value of

$$\sum_{n=0}^{1729} (-1)^n . V_n \, ?$$

ii) Prove that for any natural number n,

$$2\left(\sqrt{n+1}-1\right) < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$

2+3

b) i) If $n = p_1 p_2 ... p_k$, where $p_1, p_2, ..., p_k$ are distinct primes, prove that

$$\sum_{d|n} \left| \mu(d) \right| = 2^k.$$

ii) If $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$, where $p_1, p_2, ..., p_k$ are distinct primes and $\alpha_i \geq 1$, prove that

$$\sum_{d|n} \mu(d)\phi(d) = (2-p_1)(2-p_2)...(2-p_k).$$

2+3

c) i) Prove that $2^n 3^{2n} - 1$ is always divisible by 17.

ii) Prove that for every integer n, $n^3 \pmod 6 = n \pmod 6$. 2+3

d) i) If there is no odd prime divisor of r, prove that there is only one Pythagorean triangle within radius r.

ii) Apply Descarte's rule of signs to find the nature of the roots of the equations $x^8 + 1 = 0$. 3+2

e) i) Let A be an $m \times n$ matrix. Prove that there exists a sequence of elementary row operations of types 1 and 3 that transforms A into an upper triangular matrix.

ii) Let A be an $m \times n$ matrix. Prove that if c is any non-zero scalar, then $\text{rank}(cA) = \text{rank}(A)$. 2+3

f) i) Let B be an $n \times m$ matrix with rank m. Prove that there exists an $m \times n$ matrix A such that $AB = I_m$.

ii) If α, β, γ are the roots of the equation $x^3 - qx + r = 0$ find the relation between r and q so that

$$(\alpha-\beta)^2, (\beta-\gamma)^2, (\gamma-\alpha)^2$$

are in geometric progression. 2+3

3. Answer any **two** questions: 10×2=20

a) i) For any set A, finite or infinite, let B^A be the set of all functions from A into the set $B = \{0, 1\}$. Show that the cardinality of B^A is the same as the cardinality of the set 2^A.

ii) If p is a prime, prove that

$$2p(p-3)!+1 \equiv 0 \pmod{p}.$$

iii) Prove that 3, 5 and 7 are the only three consecutive odd integers which are prime. 3+3+4

b) i) Let f and g belong to S_n. Prove that fg is even if and only if f and g are both even or both odd.

ii) In S_3, find elements α and β such that $|\alpha| = 2$, $|\beta| = 2$ and $|\alpha\beta| = 3$.

iii) In S_4, find a cyclic subgroup of order 4 and a noncyclic subgroup of order 4.

3+3+4

c) i) Suppose that H is a subgroup of S_n of odd order. Prove that H is a subgroup of A_n.

ii) Show that the set $\{5, 15, 25, 35\}$ is a group under multiplication modulo 40. What is the identity element of this group? Can you see any relationship between this group and $U(8)$?

iii) Prove that elementary row operation of type 2 can be obtained by dividing some row by a non-zero scalar. 3+3+4

d) i) Prove or give a counter example to the following statement:

If the coefficient matrix of a system of m linear equations in n unknowns has rank m, then the system has a solution.

ii) If $(A \mid b)$ is in reduced row echelon form, prove that A is also in reduced row echelon form.

iii) Let

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ -1 & 1 & 3 & -1 & 0 \\ -2 & 1 & 4 & -1 & 3 \\ 3 & -1 & -5 & 1 & -6 \end{pmatrix},$$

find a 5×5 matrix M with rank 2 such that $AM = O$, where O is the 4×5 zero matrix. 3+3+4

—————