U.G. 2nd Semester Examination - 2022

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-03

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The symbols and notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) Construct an injective map from $\mathbb{N}$ to (0,1).

b) The Order completeness Axiom is not applicable for $\mathbb{Q}$. Justify your answer.

c) What can you say about the set A if $Sup\, A = Inf\, A$?

d) Every subset of the set of integers $\mathbb{Z}$ has a least element. Justify your answer.

e) Show that $\sum_{n=1}^{\infty} \frac{n+1}{n}$ is not convergent.

f) Let a, b be two real numbers with $a < b$. Show that there exists $r \in \mathbb{Q}$ such that $a < r\sqrt{2} < b$.

g) Show that $x^2 + 1 = 0$ has no real solution. Which property of real number ensures it?

h) Let $a_n - \frac{n!}{n^n}$. Show that $a_n \to 0$.

i) Can you construct a surjective map from $\mathbb{N}$ to (0, 1)? Justify your answer.

j) Let $|x_n| \to 2$. Does it imply $x_n \to 2$?

k) Show that Cauchy sequences are bounded.

l) Let $I_n = \left(0, \frac{1}{n}\right)$. show that $\bigcap_n I_n = \phi$.

m) Using the Sandwich lemma, prove that $\sqrt{n+1} - \sqrt{n} \to 0$.

n) Show that $Inf\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = 0$.

o) Show that the set of natural numbers $\mathbb{N}$ is unbounded.

2. Answer any **four** questions: 5×4=20

a) Show that LUB property holds iff GLB property holds true in $\mathbb{R}$.

b) Let p be any prime. Then show that there exists no rational number r such that $r^2 = p$.

c) Show that every bounded real sequence has a convergent subsequence.

d) Construct a convergent subsequence of $\{\sin n\}$.

e) Not using the Heine-Borel Theorem, show that $\{0\} \bigcup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is a compact subset of $\mathbb{R}$.

f) Show that ϕ and $\mathbb{R}$ are the only sets in $\mathbb{R}$ which are closed as well as open.

g) Using the integral test, show that $\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

3. Answer any **two** questions: 10×2=20

a) i) Prove that a set is compact in $\mathbb{R}$ iff it is closed and bounded set in $\mathbb{R}$.

ii) Show that the well-ordering is equivalent to the principle of mathematical induction.

b) Check whether the following series are convergent or not:

i) $\sum_{n=1}^{\infty} \frac{7^{n+1}}{3^{2n}}$

ii) $\sum_{n=1}^{\infty} \frac{n}{n^4 + n^2 + 1}$

c) i) Let $x_n = \sum_{n=1}^{\infty} \frac{1}{n!}$. Show that $\lim x_n$ exists and the limit is an irrational number.

ii) Show that if $\lim_{n \to \infty} a_n = a$, then

$$\lim_{n \to \infty} \left(\frac{1}{\log n}\right) \left(\sum_{k=1}^{n} \frac{a_k}{k}\right) = a.$$

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