U.G. 3rd Semester Examination - 2020

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-05

(Theory of Real Functions & Introductin to Metric Spaces)

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

Symbols and notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) Let $f(x+y)=f(x)f(y)$ for all x and y and $f(5)=-2$, $f'(0)=3$. What is the value of $f'(5)$?

b) Show that $a^x > x^a$ for $x > a \geq e$.

c) If $f(x)=2[x]+1$, find all the points of discontinuities of f in [0, 3].

d) Show that $\lim\limits_{x\to 0}\frac{|x|}{x}$ does not exist.

e) If (X, d) be metric space and x, y, z be any three points of X, prove that

$$|d(x, z)-d(y, z)| \leq d(x, y).$$

f) Prove that there exists $x \in \left(0, \frac{\pi}{2}\right)$ such that $x = \cos x$.

g) Draw open ball of unit radius about (0, 0) for the metric space $(\mathbb{R}^2, d)$, where

$$d(z_1 - z_2) = \max(|x_1 - x_2|, |y_1 - y_2|),$$

where $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are any two points of $\mathbb{R}^2$.

h) "The conditions of Rolle's theorem are only sufficient but not necessary for $f'(x)$ to vanish at some point in (a, b)" – Justify the statement with example.

i) Give example of a function which is continuous at only one point of the domain.

j) Let (X, d) be a given metric space and let (Y, d_Y) be a subspace of (X, d). Prove that every d_Y-closed subset of Y is d-closed if and only if Y is d-closed.

k) Verify the Cauchy's mean value theorem for the functions x^2 and x^3 in [1, 2].

l) Prove that $\lim_{x\to\infty} \frac{1}{x} = 0$.

m) A real function f is continuous [0, 2] and $f(0) = f(2)$. Show that there exists at least a point c in [0, 1] such that $f(c) = f(c+1)$.

n) If $f + g$ is differentiable at a point, then will f and g both be differentiable? Justify your answer.

o) Is every open set in a metric space is an open ball? Justify.

2. Answer any **four** questions: 5×4=20

a) If $f:[a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and $f(a)f(b) < 0$, then prove that there exists at least one point c in $[a, b]$ such that $f(c) = 0$.

b) Show that a subset of a metric space is open if and only if it is the union of a family of open balls.

c) Show that between any two roots of $e^x \cos x = 1$, there exists at least one root of $e^x \sin x - 1 = 0$.

d) If f is defined and continuous on $[a, b]$ and is derivable on (a, b) and if $f'(x) = 0$ for all x in (a, b), then show that $f(x)$ has a constant value throughout $[a, b]$.

e) Prove that $\frac{\tan x}{x} > \frac{x}{\sin x}$, whenever $0 < x < \frac{\pi}{2}$.

f) Let $f: I \to \mathbb{R}$ be a real function. Show that f is differentiable at a point c if and only if there exists a real function $\varphi: I \to \mathbb{R}$ that is continuous at c satisfying

$$f(x) - f(c) = \varphi(x)(x-c) \;\; \forall x \in I \text{ and } \varphi(c) = f'(c).$$

3. Answer any **two** questions from **(a)** to **(d)**: 10×2=20

a) i) Show that $f(x) = x^2$ is uniformly continuous on [–1, 1], but not uniforly continuous on $[0, \infty]$.

ii) Let (X, d) be a metric space. Show that (X, d_1) is also a metric space where $d_1(x, y) = \min\{1, d(x, y)\}$. 5+5

b) i) If φ and ψ be two functions derivable in $[a, b]$ and $\phi(x)\psi'(x) - \psi(x)\phi'(x) > 0$ for any x in this interval, then show that between two consecutive roots of $\phi(x)=0$ in $[a, b]$, there lies exactly one root of $\psi(x)=0$.

ii) Prove that every subset the discrete metric space is open as well as closed.

6+4

c) i) If $\varphi(x)=f(x)+f(1-x)$, $x\in[0, 1]$ and $f''(x)<0$ for all $x\in[0, 1]$, show that φ is increasing on $\left[0, \frac{1}{2}\right]$ and decreasing on $\left[\frac{1}{2}, 1\right]$.

ii) Show that $1-\frac{x^2}{2}\le\cos x\le 1-\frac{x^2}{2}+\frac{x^4}{24}$, for all $x\in\mathbb{R}$. 5+5

d) i) Expand $(1+x)^m$, when m is any real number, in powers of x.

ii) If f'' is continuous on some nbd of c prove that

$$\lim_{h\to 0}\frac{f(c+h)-2f(c)+f(c-h)}{h^2}=f''(c).$$

6+4

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