U.G. 3rd Semester Examination - 2021

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-05

(Theory of Real Functions & Introduction to Metric Spaces)

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

Symbols and notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

   i) If $f(x) = \frac{|x|}{x}, x \neq 0$ and $c \neq 0$ then find the value of $|f(c) - f(-c)|$.

   ii) If $f(x)$ and $g(x)$ are differentiable functions such that $f'(x) = 3x$ and $g'(x) = 2x^2$ then find $lim_{x\to 1} \frac{[(f(x)+g(x)) - (f(1)+g(1))]}{x-1}$.

   iii) Evaluate $\lim_{x\to 3}\left([x] - \left[\frac{x}{3}\right]\right)$.

   iv) Show that there exists a root of $x + xlogx - 3 = 0$ in (1,3).

   v) Examine the validity of Rolle's theorem for $f(x) = x(x+3)e^{\frac{-x}{2}}$ on [-3,0].

   vi) Find the extreme value of the function $f(x) = \frac{\log x}{x}$, in its domain.

   vii) Show that $\lim_{x\to\infty} \frac{[x]}{x} = 1$.

   viii) Examine whether the function defined by

   $$f(x) = \begin{cases} xcos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$ is differentiable at $x=0$.

   ix) Determine whether the set $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\ldots\ldots\right\}$ is open or not with respect to usual metric.

   x) Write the Statement of Darboux theorem for derivative of a function.

   xi) Determine whether the function $d(x,y) = |\sin(x-y)|$ where $x, y \in \mathbb{R}$ is a metric or not.

   xii) Let (X, d) be a discrete metric space. Prove that $\{x\}$ is an open subset of X for all $x \in X$.

   xiii) Let (X, d) be a metric space. Let $x, y \in X, x \neq y$. Prove that there exists open balls B_1 and B_2 in X so that $x \in B_1, y \in B_2$ such that $B_1 \cap B_2 = \emptyset$.

   xiv) The function defined by

   $$f(x) = \begin{cases} 2x, & x \in \mathbb{Q} \\ 1-x, & x \in \mathbb{R} - \mathbb{Q} \end{cases}.$$ Prove that $f(x)$ is continuous at $x = \frac{1}{3}$ and discontinuous at other points.

[Turn over]

xv) Show that $x < tan x$ in $0 < x < \pi/2$.

2. Answer any **four** questions: 5×4=20

a) Let $f: D \to \mathbb{R}$ be a function and a be a limit point of $D \subseteq \mathbb{R}$. Show that f is continuous at $x = c$ if and only if for every sequence $\{x_n\}$ in D converging to c, the sequence $\{f(x_n)\}$ converges to $f(c)$.

b) i) A function is defined on $\mathbb{R}$ by $f(x) = \begin{cases} 1, when & x \in \mathbb{Q} \\ 0, when & x \in \mathbb{R} - \mathbb{Q} \end{cases}$ where $\mathbb{Q}$ is set of rational numbers. Prove that f is continuous at no point $c \in \mathbb{R}$. 3

ii) Give an example with proper justifications to show that a bounded function on a closed and bounded interval need not be continuous. 2

c) If a function f be differentiable on [0, 2] and $f(0) = 0$, $f(1) = 2$, $f(2) = 1$,then prove that $f^{/}(c) = 0$ for some c in (0,2).

d) Let $f: I \to \mathbb{R}$ and $g: J \to \mathbb{R}$ be such that Image $f \subseteq$ J, where I, J are intervals in $\mathbb{R}$. Let f be differentiable at c and g be differentiable at $f(c) = d \in J$. Show that $g \circ f: I \to \mathbb{R}$ is differentiable at c and

$(g \circ f)^{/}(c) = g^{/}(c)(f(c))f^{/}(c)$.

e) If $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and is continuous at points of $\mathbb{R}$, then prove that f is uniformly continuous on $\mathbb{R}$.

f) If a function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x| + |x-1| + |x-2|$ for all $x \in \mathbb{R}$, then find derived function $f^{/}$ and specify the domain of $f^{/}$.

3. Answer any **two** questions from (**a**) to (**d**) : 10×2=20

a) i) Let $f: [a, b] \to \mathbb{R}$ be continuous. Also, let $Sup_{\, x\in[a,b]} f(x) = M$, $Inf_{\, x\in[a,b]} f(x) = m$. Show that there is at least one $c \in [a, b]$ such $f(c) = M$ and there is at least one $d \in [a, b]$ such $f(d) = m$.

ii) Show that $f(x) = \frac{1}{1+x^2}$ is uniformly continuous on $(-\infty, \infty)$. 5+5

b) i) Prove that the set $C[a, b]$ of all real valued functions continuous on the interval [a, b] with the function d defined by

$$d(f, g) = \int_a^b ((f(x) - g(x))^2)^{\frac{1}{2}} dx$$ is a metric space.

ii) Show that

$\frac{v-u}{1+v^2} < tan^{-1}v - tan^{-1}u < \frac{v-u}{1+u^2}$, if $0 < u < v$.

5+5

c) i) Find the maximum and minimum values of $y = \sin x\,(1 + \cos x)$, $0 \le x \le 2\pi$.

ii) Let a function f be twice differentiable on $[a, b]$ and $f(a) = f(b) = 0$ and $f(c) < 0$ for some c in (a, b). Prove that there exists at least one point ξ in (a, b) for which $f''(\xi) > 0$. 5+5

d) i) Find Maclaurin's infinite series expansion for the function

$f(x) = \log(1 + x), -1 < x \le -1.$ 4

ii) Use mean value theorem to prove that

$\frac{1}{x} < \frac{1}{\log(1+x)} < 1 + \frac{1}{x}.$ 3

iii) In a metric space (X, d) show that arbitrary intersection of closed set is a closed set. 3

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