U.G. 3rd Semester Examination - 2021

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-06

(Abstract Algebra)

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

Notations & Symbols have their usual meanings.

1. Answer any **ten** questions: $2\times10=20$

a) Let $GL(2, \mathbb{R})$ be the group of all 2×2 matrices over $\mathbb{R}$ under matrix multiplication and $H=\left\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : ab \neq 0\right\}$. Check whether H is a subgroup of $GL(2, \mathbb{R})$ or not.

b) Let a be an element of order 24 in a group G. Find a generator for $\langle a^9\rangle \cap \langle a^{21}\rangle$.

c) Do the odd permutations in S_n forms a group? Give reasons to your answer.

d) Let G be a group and $H=\{g^2 : g \in G\}$. Is H a subgroup of G?

e) Find the invertible elements under multiplication in $\mathbb{Z}_6$. Are they form a group?

f) Explain why $\phi(g)=3g$ is not a homomorphism from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{10}$.

g) Consider the homomorphism $\phi : S_n \to \{-1, 1\}$ by

$$\phi(\sigma)=\begin{cases} 1 \text{ if } \sigma \text{ is even} \\ -1 \text{ if } \sigma \text{ is odd}. \end{cases}$$

Find the kernel of ϕ. What can we deduce from it?

h) Let $\phi : (G, \bullet) \to (G', *)$ be a homomorphism from the group G to the group G'. Show that if H is a cyclic subgroup of G, then $\phi(H)$ is a cyclic subgroup of G'.

i) Given H and K are subgroups of a group G, Is $H \cup K$ a subgroup of G? Give reasons to your answer.

j) Show that $(\mathbb{R}\setminus\{0\}, \bullet)$ and $(\mathbb{C}\setminus\{0\}, \bullet)$ are not isomorphic.

k) Show that H is a normal subgroup of a group G if and only if for $a, b \in G$, $ab \in H$ implies $ba \in H$.

l) Let G be a group of odd order. Show that for any $x \in G$ there exists $y \in G$ such that $y^2 = x$.

m) Is $\mathbb{Z}_3 \times \mathbb{Z}_9$ is isomorphic to $\mathbb{Z}_{27}$? Give reasons to your answer.

n) Let G be a noncyclic group of order 25 and $a \in G$ is not the identity element. What is the order of a and why?

o) Let G be a group and $\phi : G \to G$ defined by $\phi(g) = g^{-1}$. Under which restrictions we can say that ϕ is an isomorphism.

2. Answer any **four** questions: 5×4=20

a) If G is a finite group with order not divisible by 3, and $(ab)^3 = a^3b^3$ for all $a, b \in G$, then show that G is abelian.

b) Let $G = \langle a \rangle$ be a group of order n. If H is a subgroup of G and order of H is m, then $H = \left\langle a^{\frac{n}{m}} \right\rangle$.

c) Let m and n be positive integers such that $m \mid n$.

i) Prove that the map $\phi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ sending $a + n\mathbb{Z}$ to $a + m\mathbb{Z}$ for any $a \in \mathbb{Z}$ is well-defined.

ii) Prove that ϕ is a group homomorphism.

iii) Prove that ϕ is surjective. 1+2+2

d) Let G be a finite group. Let S be the set of elements g such that $g^5 = e$, where e is the identity element in the group G. Prove that the number of elements in S is odd.

e) Let x, y be generators of a group G with relation $xy^2 = y^3x$, $yx^2 = x^3y$. Prove that G is the trivial group.

f) Prove that the mapping $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $(a, b) \to a - b$ is a homomorphism. What is the kernel of ϕ? Describe the set $\phi^{-1}(3)$.

3. Answer any **two** questions: 10×2=20

a) i) Let G be an abelian group and let H be the subset of G consisting of all elements of G of finite order. That is, $H = \{a \in G \mid \text{the order of } a \text{ is finite}\}$. Prove that H is a subgroup of G.

ii) Let $(\mathbb{Q}, +)$ be the additive group of rational numbers and let $(\mathbb{Q}^+, \bullet)$ be the multiplicative group of positive rational numbers. Prove that $(\mathbb{Q}, +)$ and $(\mathbb{Q}^+, \bullet)$ are not isomorphic as groups.

iii) Let G be an abelian group. Let a and b be elements in G of order m and n, respectively. Prove that there exists an element c in G such that the order of c is the least common multiple of m and n. Also determine whether the statement is true if G is a non-abelian group. 3+2+5

b) i) If order of a group G is pq, where p and q both are prime numbers. Prove that order of $Z(G)$ (center of G) is either 1 or pq.

ii) Let G and H be finite cyclic groups. Then $G \times H$ is cyclic if and only if order of G and H are relatively prime. 5+5

c) i) Suppose that G is an odd order abelian group. Show that the product of all the elements of G is the identity.

ii) Show that an infinite group must have an infinite number of subgroups.

iii) Let G be a group and $Z(G)$ is its center. For any $a \in G$, $C(a)$ is the centralizer of a. Then prove that, $Z(G) = \bigcap_{a \in G} C(a)$.

3+3+4

d) i) Let G be any cyclic group. Prove that for any subgroup H of G, the factor group G/H is cyclic.

ii) Let G be a group and let $Z(G)$ be the center of G. If $G/Z(G)$ is cyclic, then G is abelian.

iii) Let G be a group and order of G is n. Let $p \mid n$, where p is a prime number. Then G has an element of order p. 3+3+4

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