U.G. 4th Semester Examination - 2022

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-10

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The symbols and notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) Define subgroup of a group. Let, $G = (R^*, \cdot)$ (group of non-zero real number under usual multiplication) and $H = \{y \in G : y = x^n$, where n is prime or x is irrational$\}$. Check whether H is a subgroup of G or not.

b) Let $a \in G$, where G is a non-cyclic abelian group of order 4. Find order of a.

c) Determine whether $\phi : (M_2(\mathbb{R}), \cdot) \to (\mathbb{R}, \cdot)$ by $\phi(A) = det(A)$. is an isomorphism.

d) Let $\phi : (\mathbb{R}, +) \to (\mathbb{R}^{>}, .)$ defined by $\phi(r) = a^r$, where $0 < a < 1$. Is it an isomorphism? Where $\mathbb{R} >$ denotes the set if all positive real numbers.

e) If α and β are distinct 2-cycles then what are the possibilities of order $\alpha\beta$.

f) If $a, b \in G$, in a group G, then prove that ab and ba have the same order.

g) Let H be a subgroup of a group G such that $g^{-1}hg \in H$ for all $g \in G$ and all $h \in H$. Show that every left coset gH is the same as the right coset Hg.

h) Give an example of a finite noncommutative ring. Give an example of an infinite noncommutative ring that does not have a unity.

i) Show that a ring is commutative if it has the property that $ab = ca$ implies $b = c$ when $a \neq 0$.

j) Show that the set of all 2 × 2 matrices form a noncommutative ring with identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

k) Find a nonzero element in a ring that is neither a zero-divisor nor a unit.

l) Show that a commutative ring with the cancellation property (under multiplication) has no zero-divisors.

m) Show that every nonzero element of $\mathbb{Z}_n$ is a unit or a zero-divisor.

n) Prove that the intersection of any set of ideals of a ring is an ideal.

o) If an ideal I of a ring R contains a unit, show that $I = R$.

2. Answer any **four** questions: 5×4=20

a) *Let* G be a non-cyclic abelian group of order 8. If G has an element of order 4 then find the order of all elements of G.

b) Let, $(G, \cdot)$ be a group with identity e and $H = \{x^2 : x \in G\}$. Is H a subgroup of G? If not, give an example. If G is abelian, then will H be a subgroup ? 2+3

c) Let R be a ring. The center of R is the set $\{x \in R : ax = xa$ for all a in $R\}$. Prove that the center of a ring is a subring.

d) Show that a unit of a ring divides every element of the ring.

e) Define nilpotent elements in a ring. Prove that the nilpotent elements of a commutative ring form a subring.

f) Give an example of a commutative ring that has a maximal ideal that is not a prime ideal.

3. Answer any **two** questions: 10×2=20

a) i) State and prove Lagrange's theorem.

ii) Suppose H and K are subgroups of a group G such that $K \leq H \leq G$ and suppose $[H/K]$ and $[G/H]$ are both finite. then $[G/K]$ is also finite and $[G/K] = [G/H][H/K]$.

iii) Prove that every group of order p^2, where p is prime, is isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$.

3+3+4

b) i) If R is a commutative ring with unity and A is a proper ideal of R, show that R/A is a commutative ring with unity.

ii) Let R be a commutative ring with unity. Suppose that the only ideals of R are $\{0\}$ and R. Show that R is a field.

iii) List the distinct elements in the ring $\mathbb{Z}[x]/\langle 3, x^2+1 \rangle$. Show that this ring is a field.

3+3+4

c) i) Let F be an infinite field and let $f(x) \in F[x]$. If $f(a) = 0$ for infinitely many elements a of F, show that $f(x) = 0$.

ii) If I is an ideal of a ring R, prove that $I[x]$ is an ideal of $R[x]$.

iii) Give an example of a commutative ring R with unity and a maximal ideal I of R such that $I[x]$ is not a maximal ideal of $R[x]$.

3+3+4

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