U.G. 6th Semester Examination - 2022

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-13

(Metric Spaces and Complex Analysis)

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The symbols and notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) Verify whether $\dfrac{e^{\frac{4\pi i}{5}}-1}{e^{\frac{4\pi i}{5}}+1}$ is a root of the equation $(1+z)^5=(1-z)^5$.

b) Verify whether $w=z^{\frac{1}{3}}$ is algebraic function or transcendental function.

c) Using the definition of limit, verify that $\lim\limits_{z\to i}\left(z^2+2\right)=1$.

d) Show that $f(z)=\dfrac{1}{z^2}$ is uniformly continuous in $\frac{1}{2}\le|z|\le 1$.

e) Let S be a compact set of complex numbers, and let f be a continuous function on S. Show that the image of f is compact.

f) Let $f=u+iv$ be analytic in a domain D. Show that f is constant in D if Im f is constant in D.

g) Evaluate $\int_C \bar{z}\,dz$ from z = 0 to z = 4+2i along the curve C given by $z=t^2+it$.

h) Evaluate $\int_C \dfrac{dz}{z-2}$ where C is the circle $|z-1|=5$.

i) In a metric space X, show that $\bar{A}-\bar{B}\subset\overline{A-B},\ A,B\subset X$

j) Show that the set of all positive integers constitutes an incomplete metric space if $\rho(m,n)=\left|\dfrac{1}{m}-\dfrac{1}{n}\right|$.

k) Show that the set $X=\mathbb{R}$ with the metric $\rho(x,y)=\dfrac{|x-y|}{1+|x-y|}$ is bounded.

l) Show by an example that the union of an infinite number of closed sets need not be closed.

m) Prove or disprove : Every connected proper subset of R with usual metric is contained in some compact subset of $\mathbb{R}$.

n) Give example of subsets of $\mathbb{R}$ which are disjoint but not separated.

o) Let (x, d) be a metric space and A, B $\subset$ X. Show that $d(A \cup B) \le d(A) + d(A, B) + d(B)$.

2. Answer any **four** questions: 5×4=20

a) Evaluate $\int_{-2+i}^{5+3i} z^3 dz$.

b) By evaluating $\int_C e^z dz$ around the circle $|z| = 1$, show that $\int_0^{2\pi} e^{cos\theta} cos(\theta + sin\theta) d\theta = 0$.

c) If $u = (x-1)^3 - 3xy^2 + 3y^2$, determine v so that $f(z) = u + iv$ is analytic.

d) Let f be a continuous mapping of a compact metric space X into a metric space. Prove that $f(\overline{A}) = \overline{f(A)}$ for every subset A of X.

e) A sequence $\{x_n\}$ in a metric space x is convergent and converges to x. Prove that $\{x\} \cup \{x_n : n = 1, 2\}$ is a compact subset of x.

f) Let x be a metric space and let G be an open set in x. For any set $A \subset X$, prove that $G \cap A = \phi$ if and only if $G \cap \overline{A} = \phi$.

3. Answer any **two** questions: 10×2=20

a) i) Define uniform continuity of a function f(z). Show that the continuity of f(z) on a set S implies uniform continuity of f(z) on S if S is bounded and closed in $\mathbb{C}$.

ii) Show that boundedness and total boundedness are equivalent in $\mathbb{R}^n$. 5+5

b) i) Let $f(z) = e^{-z^{-4}}, z \ne 0$
$= 0, \quad z = 0$

Show that though Cauchy-Riemann equations are satisfied at (0. 0), $f'(0)$ does not exist.

ii) Prove that a sequence $x_n \to x$ in the metric space C[a,b] if and only if the corresponding sequence of real valued functions $\{x_n(t)\}$ converges uniformly to x(t) on [a, b]. 5+5

c) i) State and prove Cauchy integral formula for a function of complex variable.

ii) Prove that if f is one-one and onto continuous mapping of a compact metric space (X, d) into a metric space (Y, ρ) then f^{-1} is continuous on (Y, ρ). 5+5

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