U.G. 6th Semester Examination - 2022

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-14

(Ring Theory and Linear Algebra)

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The symbols and notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

i) If I is an ideal of a ring with unity R and $u \in I$ where u is a unit element of R, then prove that $I = R$.

ii) Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Then find a basis for the vector space

$$S = \left\{ aA^2 + bA + cI : a,b,c \in \mathbb{R} \right\}.$$

iii) Let V be a vector space and U is a subspace of V. Prove that

$$U_a = \left\{ f \in V^* : f(u) = 0 \text{ for all } u \in U \right\}$$

is a subspace of V^*, where V^* is the dual space of V.

iv) If $gcd(m,n) = 1 \left(x - \frac{m}{n} \right) | \left(a_0 + a_1 x + ... a_r x^r \right)$ where all a_i's are integers, then prove that $m \mid a_0$ and $n \mid a_r$.

v) Let $S = \{(x,y,z) \in \mathbb{R}^3 | x+y+z=k\}$ is a subspace of $\mathbb{R}^3$. Find the value of k. Then find orthogonal complement of S.

vi) Let $\lambda_1 \neq \lambda_2$ be two eigenvalues of a matrix A and u_i is an eigenvector to λ_i, $i = 1, 2$. Then show that u_1 and u_2 are linearly independent.

vii) Let the matrix A has eigen value 7 with eigen vector $v = (4, -11, 7)^T$ and $B=A-4I$. Find an eigen value and its eigen vector of B.

viii) Let $C[0,2]$, the set of all real valued continuous functions on $[0, 2]$ with inner product $\langle f, g \rangle = \int_0^2 f(x) g(x) \, dx$ for $f, g \in C[0,2]$.

Find the distance between

$f(x) = 4x + 1$ and $g(x) = \sqrt{3 - x}$.

ix) Consider an $n \times n$ matrix $A = (a_{ij})$ with $a_{12} = 1, a_{ij} = 0 \, \forall \, (i,j) \neq (1,2)$. Prove that there is no invertible matrix P such that PAP^{-1} is diagonal.

x) Let F be a field and $\phi : F \to F$ be an homomorphism. Show that ϕ is either an isomorphism or $\phi(a) = 0$ for all $a \in F$.

xi) Show that $x^3 - 9$ is reducible over $\mathbb{Z}_{11}$.

xii) Let V be the space of all $n \times n$ matrices and B be any fixed matrix in V. If f is the trace function on V and $T : V \to V$ is a linear operator defined by $T(A)=AB-BA$, then what is $T^t f$ (T^t is the transpose of T)?

xiii) Let $A = \begin{bmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix}$,

where a, b and c are real numbers. Then show that the characteristics polynomial of A is equal to the minimal polynomial for A.

xiv) If A, B are ideals in a ring R and $A \cap B = \{0\}$ Then show that for any $a \in A, b \in B, ab = 0$.

xv) Let T be the linear operator on $\mathbb{R}^2$, matrix of which (in the standard basis) is $\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$. Find the invariant subspaces of $\mathbb{R}^2$ under T.

2. Answer any **four** questions: 5×4=20

i) Let A be any 2×2 matrix over $\mathbb{C}$ and let

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

be any polinomial over $\mathbb{C}$. Show that $f(A)$ is a matrix which can be written as $c_0 I + c_1 A$ for some $c_0, c_1 \in \mathbb{C}$ whare I is the identity matrix.

ii) Let R be a Euclidean Domain and let a and b be nonzero elements of R. Let d be greatest common divisor of a and b, then the principal ideal (d) is the ideal generated by a and b.

iii) Let $a_1, a_2, \ldots, a_n$ be n distinct odd integers. Prove that the polynomial

$$f(x) = (x - a_1)(x - a_2) \cdots (x - a_n) + 1$$

is irreducible in $\mathbb{Z}[x]$.

iv) Let A be an $n \times n$ real matrix such that $A^2=I$, but $A \neq \pm I$ (Where I denotes the $n \times n$-identity matrix). Show that

a) A has two eigen values λ_1, λ_2.

b) Every element $x \in \mathbb{R}^n$ can be expressed uniquely as $x_1 + x_2$, where $Ax_1 = \lambda_1 x_1$ and $Ax_2 = \lambda_2 x_2$.

v) Let $A = (a_{ij})$ be an $n \times n$ matrix such that $a_{ij} = 0$ whenever $i \geq j$. Prove that A^n is the zero matrix.

vi) Show that every nonzero prime ideal in a Principal Ideal Domain is a maximal ideal. Hence deduce that if R is any commutative ring such that the polynomial ring $R[x]$ is a Principal Ideal Domain, then R. is necessarily a field.

vii) Let U and V be any two vector spaces over a field F and $\dim U = m$, $\dim V = n$. Show that Hom (U, V), the set of all vector space homomorphisms of U into V, is a vector space over F of dimension mn.

3. Answer any **two** questions: 10×2=20

i) a) If V is a finite dimensional inner product space and if W is a subspace of V. Then V is the direct sum of W and $W^{\perp}$, where $W^{\perp}$ is the orthogonal complement of W.

b) Show that $(W^{\perp})^{\perp} = W$.

c) Let m, n be integers such that gcd $(m,n) = 1$. Let D be an integral domain, $a, b \in D$. Suppose $a^m = b^m$ and $a^n = b^n$. Prove that $a = b$. 5+3+2

ii) Let D be a principal ideal domain.

a) Show that every element neither zero nor unit in D is a product of irreducibles.

b) An ideal $\langle p \rangle$ of D is maximal if and only if p is an irreducible.

c) For $p, a, b \in D$, where p is an irreducible. If $p|ab$, then either $p|a$ or $p|b$. 5+3+2

iii) Let V be a vector space over a field F, W is a subspace of V and $A(W) = \{f \in V^* | f(w) = 0$ for all $w \in W\}$, where V^* is the dual space of V.

a) Show that W^* is isomorphic to $V^*/A(W)$ and $\dim A(W) = \dim V - \dim W$, where W^* is the dual space of W.

b) Show that, $A(A(W)) = W$.

c) Find $A(W)$, where W = span $\{(1,2,3), (0,4,-1)\}$ is the subspace of $V = \mathbb{R}^3$.

5+2+3

iv) Let V be the inner product space of all polynomial of degree less than or equal to 3 with inner product

$$\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt \text{ for } f, g \in V.$$

Show that $\{1, x, x^2, x^3\}$ is a basis for V. Find an orthonormal basis for V by Gram-Schmidt orthogonalisation process. 10

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