U.G. 6th Semester Examination - 2021

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-14

(Ring Theory and Linear Algebra II)

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The symbols and notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

   i) Does the equation $x^2 + y^2 = 3z^2$ has a non-zero solution in Z? Justify your answer.

   ii) Determine all ring homomorphism from Z_{12} to Z_{30}.

   iii) Let F be a field. Show that the field of quotients of F is ring isomorphic to F.

   iv) Define dual basis.

   v) Let $T : R^3 \to R^3$ be the linear operator defined by $T(a,b,c) = (a+b,\ b+c,\ 0)$.

   vi) Examine whether coordinate axes and coordinate planes are T-invariant sub-spaces or not.

   vii) In $Z_3[x]$, show that the distinct polynomials $x_4 + x$ and $x_2 + x$ determine the same function from Z_3 to Z_3.

   viii) Let $x = \langle 1-i,\ 4\rangle$ and $y = \langle 2+3i,\ 4-5i\rangle$ in $\mathbb{C}^2$, find $\langle x,\ y\rangle$ and $\|x\|$.

   ix) Determine the number of irreducible polynomials over Z_p of the form $x^2 + ax + b$.

   x) Let $f(x) = x^3 + 2x + 4$ and $g(x) = 3x + 2$ in $Z_5[x]$. Determine the quotient and remainder upon dividing $f(x)$ by $g(x)$.

   xi) Does there exist any linear operator T with no T-invariant subspaces? Justify.

   xii) Show that every plane passing through the origin in $\mathbb{R}^3$ can be expressed as the null space of a vector in $\left(\mathbb{R}^3\right)^*$.

   xiii) Find a polynomial with integer coefficients that has 1/2 and –1/3 as zeros.

   xiv) Find the remainder upon dividing 98! by 101.

xv) Show that x^4+1 is irreducible over Q but reducible over R.

2. Answer any **four** questions: 5×4=20

i) Construct a field of order 25.

ii) Let $Z_3[i]=\{a+ib : a, b \in Z_3\}$. Show that the field $Z_3[i]$ is ring-isomorphic to the field $Z_3[x]/\langle x^2+1\rangle$.

iii) Prove or disprove that the field of real numbers is ring-isomorphic to the field of complex numbers.

iv) Show that the only ring automorphism of the real numbers is the identity mapping.

v) Let $P_n(R)$ be the set of all real polynomials of degree less than n, with the inner product defined by

$$\langle f(x), g(x)\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

Construct an orthonormal basis of the subspaces $P_3(R)$ from $1, x, x^2$. Hence express $f(x)=1+2x+3x^2$ as a linear combination of the above basis.

vi) In $\mathbb{C}^2$ show that $\langle x, y\rangle = xAy^*$ is in fact an inner product. Also find that $\langle x, y\rangle$ for $x=(1+i, 2-3i)$ and $y=(2-i, 3+2i)$, where $A=\begin{pmatrix} 1 & -i \\ i & 2 \end{pmatrix}$.

3. Answer any **two** questions: 10×2=20

i) Let F be a field and let a be a non-zero element of F.

a) If $a\,f(x)$ is irreducible over F, prove that $f(x)$ is irreducible over F.

b) If $f(ax)$ is irreducible over F, prove that $f(x)$ is irreducible over F.

c) If $f(x+a)$ is irreducible over F, prove that $f(x)$ is irreducible over F.

4+3+3

ii) a) Are there any non-constant polynomials in Z[x] that have multiplicative inverse? Explain your answer.

b) Let p be a prime. Are there any non-constant polynomials in $Z_p[x]$ that have multiplicative inverses? Explain your answer. 5+5

iii) a) Let V be an inner product space and W be a finite dimensional subspace of it. If $x \notin W$, prove that there exists $y \in V$ such that $y \in W^{\perp}$, but $\langle x, y \rangle \neq 0$.

b) Let $V = R^3$ and define $f_1, f_2, f_3 \in V^*$ as $f_1(x, y, z) = x - 2y$, $f_2(x, y, z) = x + y + z$, $f_3(x, y, z) = y - 3z$. Prove that $\{f_1, f_2, f_3\}$ is a basis for V^*. Also find a basis for which it is the dual basis.

5+5

iv) a) Let R be an integral domain. Prove that $\langle a \rangle = \langle b \rangle$ for some elements $a, b \in R$, if and only if $a = ub$ for some unit u of R.

b) Let R be the ring of all continuous functions on [0, 1] and let I be the collection of functions $f(x)$ in R with $f(0) = 0$. Prove that I is an ideal of R but is not a prime ideal. 5+5

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