U.G. 4th Semester Examination - 2020

MATHEMATICS

[HONOURS]

Course Code : MTMH-CC-T-10

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

Symbols and notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) What is a zero ring?

b) Give an example of a non-commutative ring with unity.

c) Find whether the set of all irrational numbers is a ring with respect to usual addition and multiplication.

d) Prove that in a commutative ring (R,+, .), $(a + b)^2 = a^2 + 2ab + b^2$ where $x^2 = x.x$, $2x = x + x$ and $a, b, x \in R$.

e) Let R be a ring with unity 1 and S be a subring of R with unity *e*. Is it necessary that 1 = *e*? Justify your answer.

f) Give an example to show that the union of two subrings of a ring may not be a subring.

g) Define characteristic of a ring R. What is the characteristic of the ring $R = (\mathbb{Z}_6,+, .)$?

h) What is an ideal of a ring?

i) Find all of the ideals of the ring $\mathbb{Z}_{18}$.

j) Which elements are divisors of zero in $\mathbb{Z}_5$ and $\mathbb{Z}_8$?

k) Define an integral domain. Give an example of a ring which is not an integral domain.

l) Evaluate $(x+\overline{1})^2$ in $\mathbb{Z}_3[x]$.

m) Define a field.

n) Give an example of a ring which is not a field.

o) Show that in a field F, $(-x)^{-1} = -x^{-1}$ where x^{-1} is the multiplicative inverse of $x \neq 0$ in F.

2. Answer any **four** questions: 5×4=20

a) Prove that the set of all even integers forms a commutative ring. 5

b) Show that the ring of matrices

$$\left\{\begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix} : a, b \in \mathbb{Z}\right\}$$

contains divisors of zero and does not contain the unity. 5

c) Prove that intersection of two ideals of a ring is an ideal of that ring. Give an example to show that union of two ideals may not be an ideal. 3+2

d) Prove that the characteristic of an integral domain is either zero or a prime number. 5

e) Prove that every field is an integral domain. Is the converse of the result true? Justify your answer. 3+2

f) Prove that the ideal $p\mathbb{Z}$ in the ring $\mathbb{Z}$ is maximal if and only if p is a prime. 5

3. Answer any **two** questions: 10×2=20

a) Define a ring with unity element. Show that the set of all rational numbers $\mathbb{Q}$ is a ring under the two compositions $\oplus$ and $\odot$ defined by $a \oplus b = a + b + 7$ and $a \odot b = a + b + \frac{ab}{7}$. If A be a ring such that $a^2 = a$ for each $a \in A$, prove that A is commutative. 2+5+3

b) Prove that a finite integral domain is a field. Discuss whether the ring of quaternions

$$H = \left\{ \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} : a, b, c, d \text{ are reals} \right\}$$

is a field. 5 + 5

c) What do you mean a Maximal ideal of a ring R? Let R be the ring of all real valued continuous functions defined on [0, 1] and let $S = \left\{ f \in R : f\left(\frac{1}{2}\right) = 0 \right\}$. Show that S is an ideal of *R*. Is *S* a maximal ideal of the ring *R*? Justify your answer. 2+5+3

d) Show that in a commutative ring *R* with unity an ideal *M* is a maximal ideal if and only if the quotient ring *R/M* is a field. Show that every maximal ideal in a commutative ring with unity is a prime ideal but converse may not be true. 6+4