U.G. 4th Semester Examination - 2021

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-8

(Riemann integration and series of functions)

Full Marks : 30 Time : $1\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The symbols and notations have their usual meanings.

1. Answer any **five** questions: 2×5=10

a) Give an example of discontinuous function defined on [0,1], which is Riemann integrable on [0,1].

b) A function *f* is defined on [0,1] by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that *f* is not Riemann integrable on [0,1].

c) Show that $\int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}$.

d) Show that $\int_0^\infty \frac{sinx}{1+x^2}$

is absolutely convergent.

[Turn Over]

e) For each natural number n, let $f_n : \mathbb{R} \mapsto \mathbb{R}$ be defined by $f_n(x) = \frac{nx}{1+n^2x^2}$, $x \in \mathbb{R}$.

Find the pointwise limit function.

f) Find the value of $\int_0^2 \frac{1}{\sqrt{x(2-x)}}$.

g) Find the radius of convergence of the power series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$.

h) Let *f* be a bounded function on [*a,b*] and *P* be any partion of [*a,b*]. Then show that $L(P, f) \leq U(P, f)$.

2. Answer any **two** questions: 5×2=10

a) Prove that $\frac{\pi^2}{9} < \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{x}{sinx}dx < \frac{2\pi^2}{9}$. 5

b) Show that $\lim_{x\to 0} \frac{\int_0^{x^2} sin\sqrt{t}dt}{x^3} = \frac{2}{3}$. 5

c) Let a function *f* is defined on [0, 1] by

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } \frac{1}{n+1} < x \leq \frac{1}{n} \ (n = 1, 2, 3, \cdots) \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that *f* is Riemann integrable on [0, 1]. Evaluate $\int_0^1 f(x)dx$. 3+2

d) Prove that

$$\int_0^{\frac{\pi}{2}} sin^p x dx \times \int_0^{\frac{\pi}{2}} sin^{p+1} x dx = \frac{\pi}{2(p+1)}, \ p > -1.$$

5

e) Assuming the power series expansion for $(1+x^2)^{-1}$ as

$(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \cdots$

obtain the power series expansion for $tan^{-1}x$ and hence deduce that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}.$$ 5

3. Answer any **one** question: 10×1=10

a) i) Prove that the series $\sum_{n=1}^{\infty}(-1)^n x^n(1-x)$ converges uniformly on [0, 1], but the series $\sum_{n=1}^{\infty} x^n(1-x)$ is not uniformly convergent on [0, 1]. 3+2

ii) Prove that a sequence of functions f_n is uniformly convergent on [a,b] to a function f if and only if $\lim_{n\to\infty} M_n = 0$, where $M_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$ and use this to examine the uniform convergence of

$f_n(x) = \frac{x}{n+x^2}$, $n = 1, 2, 3, \ldots$ and $x \in [0, 1]$. 3+2

b) i) Find a Fourier series expansion of $f(x) = 2x - x^2$ in $(0,3)$ and hence deduce that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\infty = \frac{\pi^2}{12}.$$ 5

ii) Prove that

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{cos(nx)}{n^2}, \ -\pi < x < \pi.$$ 5

c) i) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series and let $\mu = \limsup |a_n|^{\frac{1}{n}}$. If $0 < \mu < \infty$, prove that the series is absolutely convergent for $|x| < \frac{1}{\mu}$ and is divergent for $|x| > \frac{1}{\mu}$. 5

ii) Let $f_n(x) = nx(1-x)^n$ when $x \in [0,1]$. Show that the sequence of function $f_n(x)$ is not uniformly convergent on [0, 1]. 5

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