U.G. 5th Semester Examination - 2021

MATHEMATICS

[HONOURS]

Discipline Specific Elective (DSE)

Course Code : MATH-H-DSE-T-2B

(Differential Geometry)

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks. Candidates are required to give their answers in their own words as far as practicable.

1. Answer any **ten** questions: 2×10=20

   a) Consider the co-ordinate transformation

   $T: x^1 = u^1 \cos u^2,\ x^2 = u^1 \sin u^2, x^3 = u^3$

   verify whether the inverse transformation of T exists.

   b) Find the curvature of the curve

   $\gamma(\theta) = (a\cos\theta, a\sin\theta, b\theta),\ -\infty < \theta < \infty,$

   where a, b are constants.

   c) Let γ be a unit speed curve in R^3 with constant curvature and zero torsion. Show that γ is a circle.

   d) Define orientable surface. Give an example of a surface which is not orientable.

   e) Find the Gaussian curvature of the surface σ given by $\sigma(u, v) = (\cos v, \sin v, u)$.

   f) State Euler's Theorem for curves on surfaces. Mention its importance.

   g) Define umbilic points of a surface. Give an example of a surface where every point is umbilic point.

   h) When a surface is called flat? Give an example of flat surface.

   i) Define Christoffel symbols of first kind. What are their values in Euclidean space?

   j) Show that a curve on a surface is a geodesic if and only if its geodesic curvature is zero everywhere.

   k) When a surface is called minimal? Give an example.

   l) Show that any normal section of a surface is a geodesic.

   m) What do you mean by isometry? Give examples of two surfaces which are isometric.

n) Define lines of curvature. Give an example of a surface on which any curve is a line of curvature.

o) Define conjugate directions. State a necessary and suifficient condition for parametric curves to have conjugate directions.

2. Answer any **four** questions: 5×4=20

a) Prove that the curve obtained by the intersection of the cylinders $F: y = x^2$ and $G: z = x^3$ is regular. Hence obtain its parametric equation.

b) If $\vec{r} = (3t, 3t^2, 2t^3)$, is a space curve, prove that the curve is a helix.

c) Find the first fundamental form and unit surface normal vector for the surface Z = xy.

d) Find the Gaussian curvature of a surface

$$Z = \frac{1}{4}(x^2 - y^2) \text{ at } (2, 0, 1).$$

e) Find curvature, torsion and equation of osculating plane of a space curve

$$\vec{r}(t) = (3\cos t, 3\sin t, 4t) \text{ at } t = \frac{\pi}{4}.$$

f) Define developable surface. Check whether the surface $\vec{r}(u,v) = (f_1(u), f_2(u))$ is developable or not.

3. Answer any **two** questions: 10×2=20

a) i) If the vector equation of a curve is given by $\vec{r} = r(t)$, prove that its curvature

$$k = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3} \text{ and torsion } \tau = \frac{|\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}|}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2}.$$

ii) Define involute and evolute of space curve with examples. Find the equation of involutes of a given curve $\vec{r} = r(s)$.

b) i) Find the area on a surface $\vec{r} = (u\cos v, u\sin v, u)$ within the patch u=1 to u = 3 and v = 0 to $\frac{\pi}{4}$.

ii) If k is curvature of a curve on a surface whose normal curvature and the geodesic curvatures are k_n and k_g then prove that $k^2 = k_n^2 + k_g^2$.

c) i) Compute the second fundamental form of the elliptic paraboloid $r(u,v) = (u, v, u^2 + v^2)$.

ii) Show that z = f(x, y) where f is a smooth function of two variables, is a minimal surface if and only if

$$(1 + f_x^2) f_{yy} - 2 f_x f_y f_{xy} + (1 + f_y^2) f_{xx} = 0.$$

d) i) Show that the geodesic curvature of the curve $u = c$ with the metric

$$\lambda^2 (du)^2 + \mu^2 (dv)^2 \text{ is } \frac{1}{\lambda\mu}\frac{\partial\mu}{\partial u}.$$

ii) Find the differential equation of the geodesic for the metric

$$ds^2 = (du)^2 + (v^2 - u^2)(dv)^2.$$
