U.G. 3rd Semester Examination - 2022

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-05

(Theory of Real Functions & Vector Functions)

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

Symbols and notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

   i) Find $\lim_{x\to 0} e^x \, sgn(x+[x])$, where the signum function is defined as

$$sgn(x)=\begin{cases}\frac{|x|}{x}, & if\ x\neq 0\\ 0, & if\ x=0\end{cases}.$$

   ii) A function $f:\mathbb{R}\to\mathbb{R}$ is continuous on $\mathbb{R}$ and $f(x)=0, \forall x\in\mathbb{Q}$. Prove that $f(x)=0, \forall x\in\mathbb{R}$.

   iii) If $f(x)=\max\{x, x^{-1}\}$,, for x > 0, then obtain the value of $f(c).f(c^{-1})$ at $c>0$.

   iv) Let a function $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases} x^2, & if\ x\in\mathbb{Q} \\ 0, & if\ x\in\mathbb{R}/\mathbb{Q} \end{cases}.$$

Then show that $f'(0)=0$.

v) If $f(x)=-\sqrt{1-x^2}$, then show that $f'(x)$ assumes every real value as x runs over $(-1,1)$.

vi) Give an example of a function which satisfies all the conditions of Rolle's theorem, where the derivative of the function vanishes at two distinct interior points.

vii) Let $f(x)=\begin{cases} x\log x, & for\ 0<x\le 1 \\ 0, & for\ x=0. \end{cases}$

Is there any subinterval of $[0,1]$, where f is monotone increasing? Give reasons to support your answer.

viii) Deduce Lagrange's mean value theorem from Cauchy mean value theorem.

ix) Using Mean value theorem prove that

$$0<\frac{1}{x}\log\frac{e^x-1}{x}<1, \forall x>0.$$

x) Let $c\in\mathbb{R}$ and a real function f be such that f'' is continuous on some nbd of c. Prove that

$$\lim_{x\to 0}\frac{f(c+h)-2f(c)+f(c-h)}{h^2}=f''(c).$$

xi) If f is derivable on $[0,1]$, then show by Cauchy's mean value theorem that $f(1)-f(0)=\frac{f'(x)}{2x}$ has at least one solution in $(0,1)$.

xii) A particle moves along the curve $x=e^{-t}, y=2\cos 3t, z=2\sin 3t$. Determine the velocity and acceleration at any time t and the magnitudes at $t=0$.

xiii) Prove that $\bar{\nabla}(\bar{r}.\bar{a})=\bar{a}$, where $\bar{a}$ is the constant vector and $\bar{r}$ is the position vector of a point.

xiv) Determine the constants a, b, c so that the vector

$$\bar{A}=(x+2y+az)i+(bx-3y-z)j+(4x+cy+2z)k$$

is irrotational.

xv) If $\bar{A}=2t\,i+3t^2j-(4t+1)k$ and $\bar{B}=t\,i+3\,j+t^2\,k$, then find the value of $\int_0^2 \bar{A}\times\bar{B}\,dt$.

2. Answer any **four** questions: 5×4=20

a) i) Let a function $f:\mathbf{R}\to\mathbf{R}$ be defined by

$$f(x)=\begin{cases} x, if\ x\in \mathbf{Q} \\ 2-x, if\ x\in \mathbf{R}/\mathbf{Q}. \end{cases}$$

Then show that lim $\lim_{x\to 1} f(x)=1$ and $\lim_{x\to c} f(x)$ does not exist if $c\neq 1$.

ii) Let $\lim_{x\to c} f(x) = l(\neq 0)$. Then show that $\exists$ a $\delta > 0$ for which $\frac{3}{2}|l| > |f(x)| > \frac{1}{2}|l|$. whenever $0 < |x-a| < \delta$. 3+2

b) i) The functions $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ both are continuous on $\mathbb{R}$. Then show that the set $S = \{x \in \mathbb{R}: f(x) = g(x)\}$ is a closed set in $\mathbb{R}$.

ii) Construct a function $f: [0, \frac{\pi}{2}] \to \mathbb{R}$ which is unbounded on the closed interval and discontinuous only at the point $\frac{\pi}{2}$. 4+1

c) i) If $f(x) = \begin{cases} 2x+3, if\ x \leq 1 \\ ax^2 + bx, if\ x > 1 \end{cases}$' where a and b are real constants and if f is derivable everywhere, then find the values of a and b.

ii) Show that $f(x) = [x]$ in $[0,2]$ is not derivable of any function. 3+2

d) i) Let $f: [a,b] \to \mathbb{R}$ be a thrice differentiable function on $[a, b]$. If $f(a) = f(b) = f'(a) = f'(b) = 0$, then prove that $f'''(c) = 0$ for some $c \in (a,b)$.

ii) Show that between any two roots of $e^x \sin x = 1$, , there is a real root of $e^x \cos x + 1 = 0$. 3+2

e) If f is derivable in $[a,b]$ and $f'(a)$ and $f'(b)$ are of opposite signs, then show that there exists at least one point $c \in (a,b)$ such that $f'(c)=0$. 5

f) A function $f:\mathbb{R} \to \mathbb{R}$ is continuous and $f(x+y) = f(x).f(y)\ \forall x,y \in \mathbb{R}$. If $f(1)=c$ then show that $f(x)=cx, \forall x \in \mathbb{R}$. 5

3. Answer any **two** questions: 10×2=20

a) i) If $f:[0,1] \to \mathbb{R}$ is such that $f''(x) < 0$ in $[0,1]$ and if $\phi(x) = f(x) + f(1-x)$ in $[0,1]$, show that ϕ is monotone increasing in $\left[0,\frac{1}{2}\right]$ and it is monotone decreasing in $\left[\frac{1}{2},1\right]$.

ii) Obtain the Maclaurin's infinite series expansion of $(1+x)^m$, where m is any real number other than positive integer and $|x|<1$. 5+5

b) i) Let $D \subset \mathbb{R}$ and $f: D \to \mathbb{R}$ be a function. Let c be a limit point of D and $l \in \mathbb{R}$. Then prove that $\lim_{x \to c} f(x) = l$ iff for every sequence $\{x_n\}$ in $D-\{c\}$ converging to c, the sequence $\{f(x_n)\}$ converges to l.

ii) Show that

$$F = (x + 2y + 4z)i + (2x - 3y - z)j + (4x - y + 2z)k$$

can be expressed as the gradient of a scalar function.

5+5

c) i) If $f(x) = \sin x$, then prove that $\lim_{h \to 0^+} \theta = \frac{1}{\sqrt{3}}$, where θ is given by

$$f(h) = f(0) + hf'(\theta h), 0 < \theta < 1.$$

ii) Use Cauchy's mean value theorem to evaluate

$$\lim_{h \to 1} \frac{\cos \frac{1}{2}\pi x}{\log\left(\frac{1}{x}\right)}.$$

iii) If $\bar{w}$ is a constant vector, $\bar{r}$ and $\bar{s}$ are vectors functions of a scalar variable t and if $\frac{d\bar{r}}{dt} = \bar{w} \times \bar{r}$, $\frac{d\bar{s}}{dt} = \bar{w} \times \bar{s}$, then show that $\frac{d}{dt}(\bar{r} \times \bar{s}) = \bar{w} \times (\bar{r} \times \bar{s})$.

4+3+3

d) i) A function f is defined on $[0,1]$ by $f(0) = 1$ and 0, *if x is rational*

$$f(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \end{cases}$$ *where* m, n *are positive integers prime to each other.*

Prove that f is continuous at every irrational point in $[0,1]$ and discontinuous at every rational point in $[0,1]$.

ii) Find the maximum value of the directional derivative of f $f(x,y,z)=2xz-y^2$ at the point (1,3,2) and also find the direction in which it occurs.

iii) Give an example with proper justifications to show that a function $f(x)$ has no extreme value at $x=c$ but $f'(c)=0$.

5+3+2
