## U.G. 3rd Semester Examination - 2022 MATHEMATICS

## [HONOURS]

Course Code: MATH-H-CC-T-05

(Theory of Real Functions & Vector Functions)

Full Marks: 60

Time:  $2\frac{1}{2}$  Hours

The figures in the right-hand margin indicate marks.

Symbols and notations have their usual meanings.

1. Answer any ten questions:

 $2 \times 10 = 20$ 

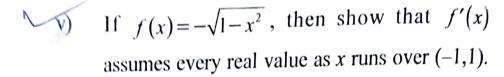
i) Find  $\lim_{x\to 0} e^x sgn(x+[x])$ , where the signum function is defined as

$$sgn(x) = \begin{cases} \frac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

- ii) A function  $f: \mathbb{R} \to \mathbb{R}$  is continuous on  $\mathbb{R}$  and  $f(x) = 0, \forall x \in \mathbb{Q}$ . Prove that  $f(x) = 0, \forall x \in \mathbb{R}$ .
- iii) If  $f(x) = \max \{x, x^{-1}\}$ , for x > 0, then obtain the value of  $f(c) \cdot f(c^{-1})$  at c > 0.
- iv) Let a function  $f: \mathbb{R} \to \mathbb{R}$  be defined by

$$f(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R}/\mathbb{Q} \end{cases}$$

Then show that f'(0) = 0.



vi) Give an example of a function which satisfies all the conditions of Rolle's theorem, where the derivative of the function vanishes at two distinct interior points.

Let 
$$f(x) = \begin{cases} x \log x, \text{ for } 0 < x \le 1 \\ 0, \text{ for } x = 0. \end{cases}$$

Is there any subinterval of [0,1], where f is monotone increasing? Give reasons to support your answer.

- viii) Deduce Lagrange's mean value theorem from Cauchy mean value theorem.
- ix) Using Mean value theorem prove that  $0 < \frac{1}{x} log \frac{e^x 1}{x} < 1, \forall x > 0.$
- Let  $c \in \mathbb{R}$  and a real function f be such that f'' is continuous on some nbd of c. Prove that  $\lim_{x \to 0} \frac{f(c+h) 2f(c) + f(c-h)}{h^2} = f''(c).$

xi) If f is derivable on [0,1], then show by Cauchy's mean value theorem that  $f(1) - f(0) = \frac{f'(x)}{2x}$  has at least one solution in (0,1).

Xii) A particle moves along the curve  $x = e^{-t}$ ,  $y = 2\cos 3t$ ,  $z = 2\sin 3t$ . Determine the velocity and acceleration at any time t and the magnitudes at t = 0.

Prove that  $\nabla(\bar{r}.\bar{a}) = \bar{a}$ , where  $\bar{a}$  is the constant vector and  $\bar{r}$  is the position vector of a point.

Determine the constants a, b, c so that the vector  $\bar{A} = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$  is irrotational.

If  $\overline{A} = 2t i + 3t^2 j - (4t + 1)k$  and  $\overline{B} = t i + 3 j + t^2 k$ , then find the value of  $\int_0^2 \overline{A} \times \overline{B} dt.$ 

Answer any four questions:

 $5 \times 4 = 20$ 

a) Let a function  $f: \mathbb{R} \to \mathbb{R}$  be defined by

$$f(x) = \begin{cases} x, if \ x \in \mathbb{Q} \\ 2 - x, if \ x \in \mathbb{R}/\mathbb{Q}. \end{cases}$$

Then show that  $\lim_{x\to 1} f(x) = 1$  and  $\lim_{x\to c} f(x)$  does not exist if  $c \neq 1$ .

Let 
$$\lim_{x\to c} f(x) = l(\neq 0)$$
. Then show that

 $a \delta > 0$  for which  $\frac{3}{2}|l| > |f(x)| > \frac{1}{2}|l|$ . whenever  $0 < |x-a| < \delta$ .

The functions  $f: \mathbb{R} \to \mathbb{R}$  and  $g: \mathbb{R} \to \mathbb{R}$ b) both are continuous on  $\mathbb R$ . Then show that the set  $S = \{x \in \mathbb{R}: f(x) = g(x)\}$  is a closed set in  $\mathbb{R}$ .

> Construct a function  $f: [0, \frac{\pi}{2}] \to \mathbb{R}$  which ii) is unbounded on the closed interval and discontinuous only at the point  $\frac{\pi}{2}$ . 4+1

(i) If  $f(x) = \begin{cases} 2x + 3, & \text{if } x \le 1 \\ ax^2 + bx, & \text{if } x > 1 \end{cases}$ where a and b are real constants and if f is derivable everywhere, then find the values of a and b.

d)

i)

Show that f(x) = [x] in [0,2] is not derivable of any function. 3+2Let  $f: [a, b] \to \mathbb{R}$  be a thrice differentiable

function on [a, b]. If f(a) = f(b) = f'(a) = f'(b) = 0, prove that f'''(c) = 0 for some  $c \in (a,b)$ .

Show that between any two roots of  $e^x \sin x = 1$ , there is a real root of  $e^x \cos x + 1 = 0.$ 3+2

435/Math  $\frac{2x+3-5}{2x-2} \underbrace{2(x-1)}_{2}$ 22-1 = x+1=2

If f is derivable in [a,b] and f'(a) and f'(b)are of opposite signs, then show that there exists at least one point  $c \in (a, b)$  such that f'(c) = 0.

 $f: \mathbb{R} \to \mathbb{R}$  is continuous and  $f(x+y) = f(x).f(y) \ \forall x,y \in \mathbb{R}.$  If f(1) = cthen show that  $f(x) = cx, \forall x \in \mathbb{R}$ . 5

- Answer any two questions: 3.  $10 \times 2 = 20$ 
  - If  $f:[0,1] \to \mathbb{R}$  is such that f''(x) < 0 in i) a) [0,1] and if  $\phi(x) = f(x) + f(1-x)$  in [0,1], show that  $\phi$  is monotone increasing in  $\left[0,\frac{1}{2}\right]$  and it is monotone decreasing in  $\left[\frac{1}{2},1\right]$ .
    - Obtain the Maclaurin's infinite series expansion of  $(1 + x)^m$ , where m is any real number other than positive integer and 5+5 |x| < 1.
- Let  $D \subset \mathbb{R}$  and  $f:D \to \mathbb{R}$  be a function. Let b) **i**) c be a limit point of D and  $l \in \mathbb{R}$ . Then prove that  $\lim_{x\to c} f(x) = l$  iff for every 2'E3 L-6 sequence  $\{x_n\}$  in  $D-\{c\}$  converging to c, the sequence  $\{f(x_n)\}\$  converges to I.

435/Math 4(2+1) = 4(2)/4(1) 4'(2+1) = 4'(2)/6  $\cos x + e^{-2} = 0$  4(2+1) = 4(2)/6

(ii) Show that

$$\bar{F} = (x + 2y + 4z)i + (2x - 3y - z)j + (4x - y + 2z)k$$
 can be expressed as the gradient of a scalar function. 5+5

- (c) i) If  $f(x) = \sin x$ , then prove that  $\lim_{h \to 0^+} \theta = \frac{1}{\sqrt{3}}$ , where  $\theta$  is given by  $f(h) = f(0) + hf'(\theta h), 0 < \theta < 1$ .
  - ii) Use Cauchy's mean value theorem to evaluate

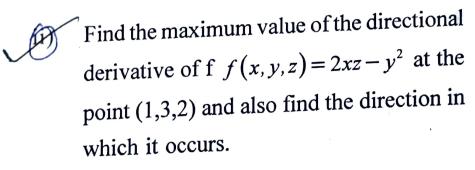
$$\lim_{h\to 1}\frac{\cos\frac{1}{2}\pi x}{\log\left(\frac{1}{x}\right)}.$$

- Vectors functions of a scalar variable t and if  $\frac{d\vec{r}}{dt} = \vec{w} \times \vec{r}$ ,  $\frac{d\vec{s}}{dt} = \vec{w} \times \vec{s}$ , then show that  $\frac{d}{dt}(\vec{r} \times \vec{s}) = \vec{w} \times (\vec{r} \times \vec{s})$ .
- d) i) A function f is defined on [0,1] by f(0) = 1 and  $[0, if \times is \ rational]$

$$f(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \\ & \text{where } m, n \text{ are} \end{cases}$$

positive integers prime to each other.

Prove that f is continuous at every irrational point in [0,1] and discontinuous at every rational point in [0,1].



iii) Give an example with proper justifications to show that a function f(x) has no extreme value at x = c but f'(c) = 0.

5+3+2