U.G. 3rd Semester Examination - 2022

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-06

(Group Theory-I)

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

Symbols and notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) If a and b are elements of a group G with the identity element e such that $ab \neq ba$, then prove that $aba \neq e$.

b) If a be an element of a group of order n and p is prime to n, then prove that the order of a^p is n.

c) Let GL(2, $\mathbb{R}$) be the group of all non-singular 2×2 matrices over $\mathbb{R}$. Prove that the set

$$H = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2 + b^2 \neq 0 \right\}$$ is a subgroup of

GL(2, $\mathbb{R}$).

d) If in a group G, $ba = a^m b^n$ for all $a, b \in G$, show that $O(a^m b^{n-2}) = O(ab^{-1})$, where m, n are integers.

$a^{m-1} b^{n-1} = \frac{a^m b}{ab} = \cancel{\frac{ba}{ab}}$

[Turn over]

e) Let a and b be two elements of a group G. Show that there exists an element $g \in G$ such that $g^{-1}abg = ba$.

f) Let a and b be two elements of a group such that $O(a) = 4$, $O(b) = 2$, and $a^3b = ba$. Show that $O(ab) = 2$.

g) Give an example of a finite abelian group which is not cyclic.

$a^3b = b$

h) Prove that $\left\{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\right\}$ is a cyclic subgroup of GL(2, $\mathbb{R}$).

i) If $P = \left\{\dfrac{1+3n}{1+3m} : m, n \in \mathbb{Z}\right\}$, then show that P is a subgroup under multiplication of all non-zero rational numbers.

j) Verify whether

$$\left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{R} \text{ and } ac \neq 0\right\}$$ is a normal subgroup of GL(2, $\mathbb{R}$).

k) Show that the group ($\mathbb{Q}$, +) is not cyclic.

l) Give an example of an infinite group each element of which has a finite order.

m) Let K be a subgroup of a group G such that $x^2 \in K$ for all $x \in G$. Prove that K is normal in G.

n) Show that $(\mathbb{Q}, +)$ is not isomorphic to $(\mathbb{Q}^+, +)$.

o) Let G be a group. Prove that the mapping $\alpha(g) = g^{-1}$ for all $g \in G$ is an automorphism if and only if G is abelian.

$\phi(a) = a^{-1}$ $ab = ba$

2. Answer any **four** questions: 5×4=20

a) Let H be a subgroup of a finite group G. Suppose that $g \in G$ and n is the smallest positive integer such that $g^n \in H$. Prove that n divides O(G).

$\phi(ab) = (ab)^{-1} = b^{-1}a^{-1} = \phi(ba)$

b) If H and K are subgroups of a group G, then prove that $H \cup K$ is a sub-group of G if and only if $H \subseteq K$ or $K \subseteq H$.

15
20
10
45

c) Let H be a subgroup of a group G. Prove that

$$\bigcap_{g \in G} gHg^{-1}$$ is a normal subgroup of G.

d) Prove that the additive group G of complex numbers $a+ib$ is isomorphic to the multiplicative group G' of rationals of the form $2^a 3^b$ $(a, b \in \mathbb{Z})$.

e) Prove that any finitely generated subgroup Of $(\mathbb{Q}, +)$ is cyclic.

f) Let $f : G \to G_1$ be a homomorphism of groups. Then the quotient group $G/Ker(f)$ is isomorphic to the subgroup $Im(f)$ of G_1.

3. Answer any **two** questions: $10 \times 2 = 20$

a) i) Prove that a finite group can not be expressed as the union of two of its proper subgroups. 4

ii) Prove that every group of prime order is cyclic. 3

iii) Let G be a subgroup of a group G such that $[G : H] = 2$. Prove that H is a normal subgroup of G. 3

b) i) Let H and K be subgroups of a finite group G with $H \subseteq K \subseteq G$. Prove that $[G{:}H] = [G{:}K]\,[K{:}H]$. 4

ii) Prove that the nth roots of unity form a cyclic group. 3

iii) Let ρ be a congruence relation on a group G. Show that there exists a normal subgroup H of G such that $\rho = \{(a, b) \in G \times G : a^{-1}b \in H\}$. 3

c) i) Let G be a group and $Z(G)$ be the center of G. If $G/Z(G)$ is cyclic, then prove that G is abelian. 3

ii) Prove that a subgroup H of a group G is a normal subgroup if and only if every right coset of H is also a left coset. 3

iii) Let G, H, and K be groups. Suppose that the mappings $f : G \to H$ and $g : H \to K$ are homomorphisms. Prove that $gf : G \to K$ is also a homomorphism.

4

d) i) Let f be a homomorphism of a group G into a group H. Then prove that f is one-one if and only if $Ker(f) = \{e\}$, where e is the identity element of the group G. 3

ii) Prove that an infinite cyclic group is isomorphic to the additive group $\mathbb{Z}$ of all integers. 4

iii) Let G be a finite group with the identity element e and f be an automorphism of G such that for all $a \in G$, $f(a) = a$ if and only if $a = e$. Show that for all $g \in G$, there exists $a \in G$ such that $g = a^{-1}f(a)$.

3

$gf(g_1)$

$g(g_1)$