Derangements

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Introduction

Problem statement

How many permutations of $[n] = \{1, 2, ..., n\}$ have no "fixed points"? Such permutations are called **"Derangements"** and number of derangements is denoted by $D(n)$ (also by $!n$)

PIE

Do you remember?

- **Principal of exclusion and inclusion (PIE):** If $A_1, A_2, ..., A_n$ are subsets of same finite set A, then

$$|A_1 \cup A_2 \cup ... \cup A_n| = \sum_{j=1}^{n} (-1)^{j-1} \sum_{i_1, i_2, .. i_j} |A_{i_1} \cap ... \cap A_{i_j}| \quad (1)$$

known as **"Sieve formula"**

Think the problem in another way

Proof

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The compliment of the given case is *at least one* of from the party has picked his own hat.

The total # ways to pick any hat is $n!$.

We denote

$A_i = \{\#$ ways so that i' th person does get his own hat$\}; \forall i \in [n]$

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$$A_i = (n-1)!, \forall i \tag{2}$$

$$\sum_{i=1}^{n} A_i = \binom{n}{1}(n-1)! = n! \tag{3}$$

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Observe we have done some over-counting in previous step. So we have to take out all of $A_i \cap A_j$

$$A_i \cap A_j = (n-2)!; \forall i \neq j \tag{4}$$

$$\sum_{i,j \in [n]} A_i \cap A_j = \binom{n}{2}(n-2)! = \frac{n!(n-2)!}{2!(n-2)!} = \frac{n!}{2!} \tag{5}$$

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Similarly

$$\sum_{i_1,i_2,\ldots,i_k\in[n]} \left\{\bigcap_{j=1}^{k} A_{i_j}\right\} = \binom{n}{k}(n-k)! = \frac{n!}{k!} \qquad (6)$$

Proof

So from (1) total, # ways of at least one of them get his own hat is

$$|A_1 \cup A_2 \cup ... \cup A_n| = \sum_{j=1}^{n} (-1)^{j-1} \sum_{i_1, i_2, ..i_j} |A_{i_1} \cap ... \cap A_{i_j}|$$

$$n! - \frac{n!}{2!} + \frac{n!}{3!} - .... + (-1)^{n-1}\frac{n!}{n!} = A(n) \qquad (7)$$

Proof

So from (1) total, # ways of at least one of them get his own hat is

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Conclusion

Final answer

The number of derangements is

$$D(n) = n! - A(n) = \sum_{i=2}^{n} (-1)^i \frac{n!}{i!}$$

□

Limiting Case

$$\frac{!n}{n!} = \frac{D(n)}{n!} = \frac{1}{0!} - \frac{1}{1!} + \sum_{i=2}^{n} (-1)^i \frac{1}{i!} = \sum_{i=0}^{n} (-1)^i \frac{1}{i!}$$

Taking $\lim_{n \to \infty}$ at both sides

$$\lim_{n \to \infty} \frac{!n}{n!} = \lim_{n \to \infty} \sum_{i=0}^{n} (-1)^i \frac{1}{i!} = e^{-1}$$

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If things are not so good, you maybe want to imagine something better — John Forbes Nash

Thank You!