Shortest Route in Rectangular Grid

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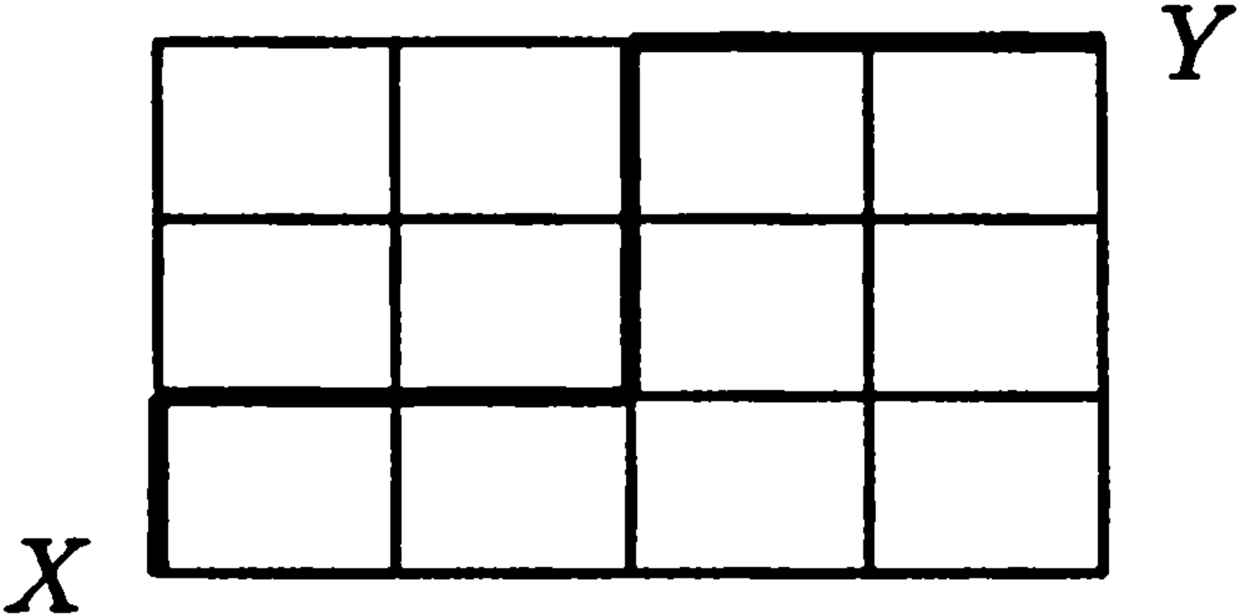
A game is played on the annual sports day of NVC, where a student has to walk either vertically or horizontally through the streets as given in the picture and wins the game if he/she walks from corner X to the corner Y using lowest number of lattice points. How many routes are available to the students?



Basically you have to find how many shortest route are there from X to Y available to the students?

Lemma

There are 2^7 binary sequences of length 7. and $\binom{7}{3}$ binary sequences containing 3 zeroes and 4 ones.

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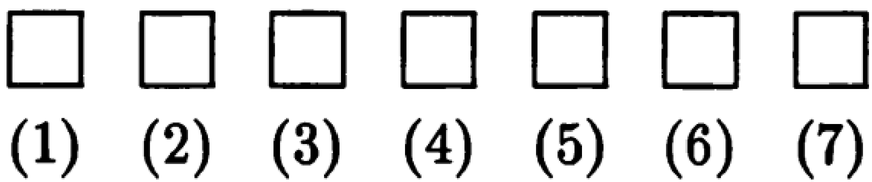

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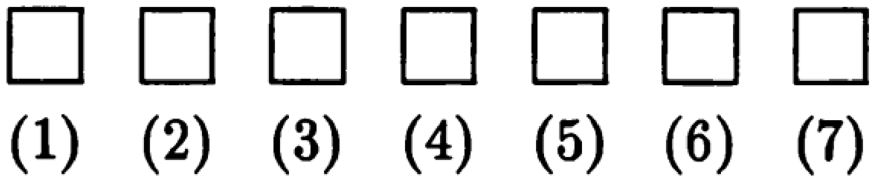

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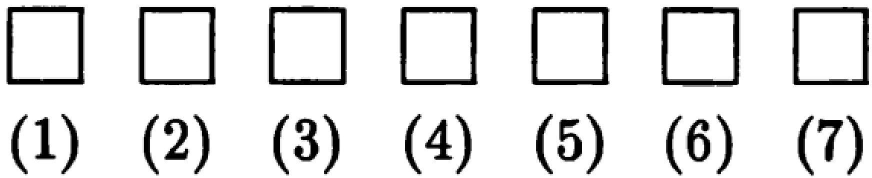

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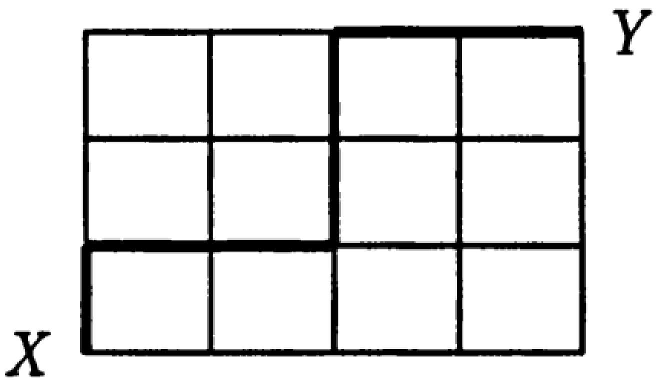
Select any 3 to put 0 - in how many ways ? $\rightarrow \binom{7}{3}$
Do we have to choose places for 1? No, because they were already chosen when we put 0's in their places. So $\binom{7}{3}$ binary sequences containing 3 zeroes and 4 ones.

Idea to shortest route problem

Let define a set $A = \{\text{all shortest route from } X \text{ to } Y\}$, we have to get the cardinality of this set.

Notice every route in A consists of 7 continuous segments of which 4 are horizontal and 3 are vertical.



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- Horizontal segment as 0
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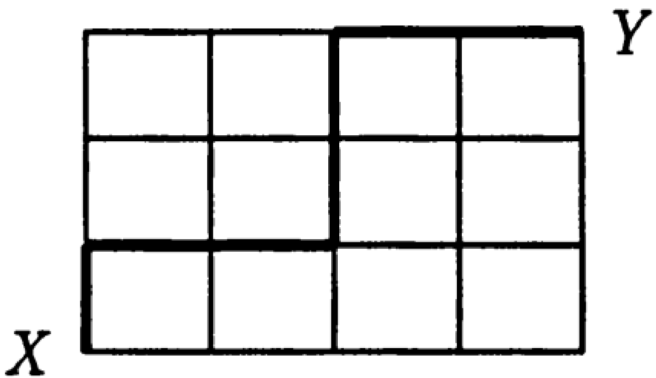

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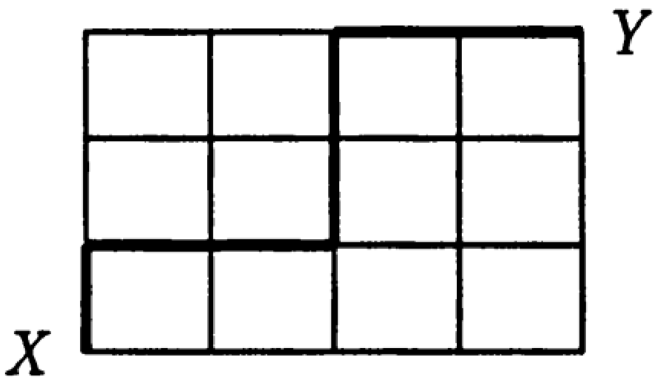
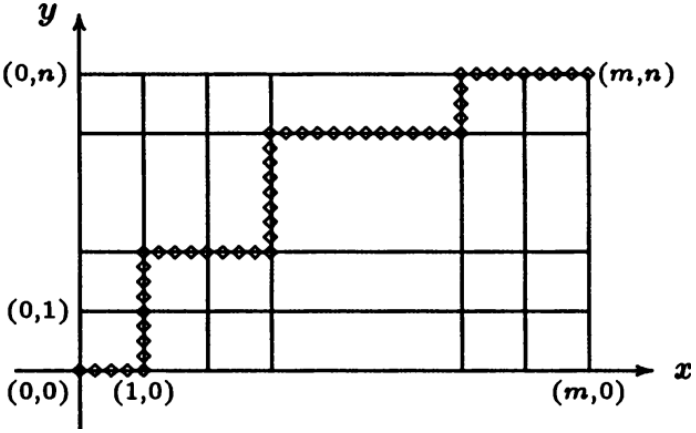


then every route in A can be uniquely represented by a binary sequence of length 7 with 4 zeroes and 3 ones. Now we can relate this problem with previous problem.
Suppose $B = \{\text{all binary sequences of length 7 with 4 zeroes and 3 ones}\}$. if there we make $f : A \to B$, then clearly it is bijection and thus $|A| = |B| = \binom{7}{3}$

Generalisation

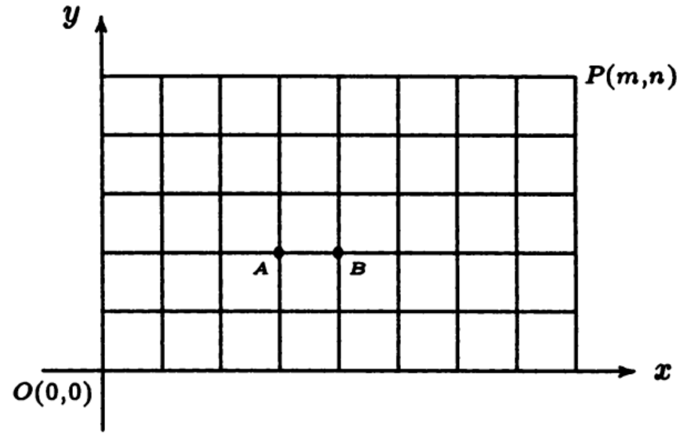
If it is an $(m+1) \times (n+1)$ grid, it has m vertical and n horizontal streets., then
# shortest route from the southwest corner X to the northeast corner Y is equal to the number of binary sequences of length $m+n$ with m zeroes and n ones is given by

$$\binom{m+n}{m} \text{ or } \binom{m+n}{n}$$



More Generalisation

Consider two lattice points $A(x, y)$ and $B(x+1, y)$



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1. # shortest routes from $O(0,0)$ to $A(x, y)$ is $\binom{x+y}{x}$ and # shortest routes from $A(x, y)$ to $P(m, n)$ is $\binom{(m-x)+(n-y)}{m-x}$. Then the number of shortest routes from $O(0,0)$ to $P(m, n)$ that pass through $A(x, y)$ is

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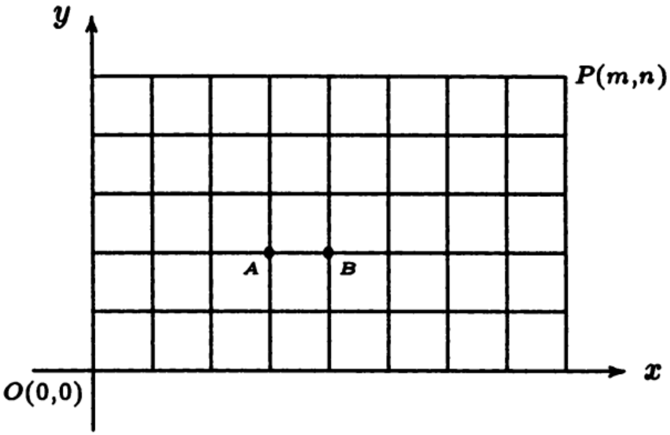

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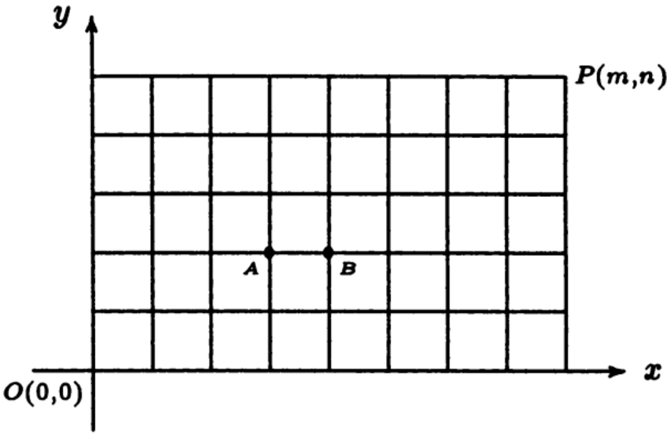

2. # shortest routes from $O(0,0)$ to $A(x, y)$ is $\binom{x+y}{x}$ and # shortest routes from $B(x+1, y)$ to $P(m, n)$ is $\binom{(m-x-1)+(n-y)}{m-x-1}$. Then the number of shortest routes from $O(0,0)$ to $P(m, n)$ that pass through the line segment AB is

$$\binom{x+y}{x}\binom{(m-x-1)+(n-y)}{m-x-1}$$

Thank You All !!