U.G. 6th Semester Examination - 2023

MATHEMATICS

[HONOURS]

Discipline Specific Elective (DSE)

Course Code : MATH-H-DSE-T-03B

(Number Theory)

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

The symbols and notations have their usual meanings.

1. Answer any **ten** questions: $2\times10=20$

   a) If n is an odd integer, show that n^4+4n^2+11 is of the form $16k$.

   b) Verify that if an integer is simultaneously a square and a cube, then it must be either of the form $7k$ or $7k+1$.

   c) Prove that the product of four consecutive integers is 1 less than a perfect square.

   d) Show that for a positive integer n and any integer a, gcd $(a, a+n)$ divides n.

e) If $\gcd(a, b) = 1$, prove that $\gcd(a^2, b^2) = 1$.

f) Using Euclidean Algorithm, obtain integers x and y such that

$$\gcd(1769, 2378) = 1769x + 2378y.$$

g) Find all prime numbers that divide $100!$.

h) Prove that the only prime of the form $n^2 - 4$ is 5.

i) Prove that if $n > 2$, then there exists a prime p satisfying $n < p < n!$.

j) For any integer a, show that $a^2 - a + 7$ ends in one of the digits 3, 7 or 9.

k) Prove that $53^{103} + 103^{53}$ is divisible by 39.

l) Solve $x^2 + 7x + 10 \equiv 0 \ (mod\ 11)$.

m) For what values of n does $n!$ exist?

n) Find a prime number p that is simultaneously expressible in the forms $x^2 + y^2$, $u^2 + 2v^2$, and $r^2 + 3s^2$.

o) Show that there are no positive integers n, such that $\sigma(n) = 10$.

2. Answer any **four** questions: 5×4=20

a) Given integers a and b, not both of which are zero, prove that there exist integers x and y such that $\gcd(a, b) = ax + by$.

b) Assuming that p_n is the nth prime number, show that the sum $\frac{1}{p_1}+\frac{1}{p_2}+...+\frac{1}{p_n}$ is never an integer.

c) Show that, the quadratic congruence $x^2 + 1 \equiv 0$ $(mod\ p)$, where p is an odd prime, has a solution if and only if $p \equiv 1\ (mod\ p)$.

d) Prove that if p is prime, then $(p-1)! \equiv -1\ (mod\ p)$.

e) Find all values of $n \geq 1$ for which $n! + (n+1)! + (n+2)!$ is perfect square.

f) State and prove the quadratic reciprocity law.

3. Answer any **two** questions: 10×2=20

a) i) Show that if p is a prime divisor of $839 = 38^2 - 5.11^2$, then $(5/p) = 1$. Use this fact to conclude that 839 is a prime number. 4

ii) For an odd prime, prove that the congruence $2x^2 + 1 \equiv 0\ (mod\ p)$ has a solution if and only if $p \equiv 1$ or $3\ (mod\ 8)$. 6

b) i) Show that for any positive integer n, $\sigma(n)$ is an odd integer if and only if n is a perfect square or twice a perfect square. 4

ii) Using Wilson's theorem, prove that for any odd prime p,

$$1^2.3^2.5^2...(p-1)^2 \equiv (-1)^{\frac{p+1}{2}} \pmod{p}.$$

6

c) i) Establish that for any positive integer n,

$$\frac{\sigma(n!)}{n} \geq 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}.$$ 5

ii) Prove that for any positive integer r, the product of any r consecutive positive integers is divisible by $r!$ 5
