NABADWIP VIDYASAGAR COLLEGE
NABADWIP, WEST BENGAL

DEPARTMENT OF MATHEMATICS

Pigeonhole Principle Talk

Departmental Seminar Day

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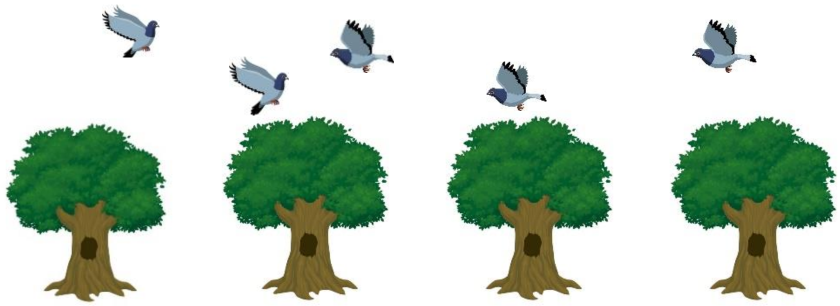


I am Arighna Pan of 6th sem.

The subject of combinatorics is so vast, where we learn how to count, find the ways of choosing. Here a basic life fact becomes very handy to solve some great problems. PHP is one of those topics in combinatorics. PHP epitomizes one of the most attractive treats of this field - the possibility of obtaining very strong results by very simple means. Look at the following statements:

- 7 is greater than 6
- 4 is greater than 3
- 10 is less than 12

These statements do not seems to be very interesting, exciting or deep. We will see! PHP makes excellent use of them.

4 trees 5 birds, where does the last bird go?

1 PP1

If n objects are placed in r boxes where $n > r$, then at least one of the boxes contains more than one object

Proof: Though it is very straight forward. Suppose N and R be two finite sets where $|N| = n > r = |R|$. Consider $f : N \to R$ then $\exists a \in R; |f^{-1}(a)| \geq 2$. In general $|f^{-1}(a)| \geq \lceil \frac{n}{r} \rceil$, Otherwise, $\forall a \in R; |f^{-1}(a)| < \frac{n}{r} \Rightarrow n = \sum_{a \in R} |f^{-1}(a)| < r \times \frac{n}{r} = n$ which is absurd.

2 PP2

For $n, t \in \mathbb{Z}^+, tn + 1$ or more objects are placed in n boxes, then at least one box gets more than t objects

3 PP3

If the average of n positive numbers is t, then at least one of the numbers is greater than or equal to t, furthermore at least one of the numbers is less or equal to t.

Proof: $\frac{\sum_{i=1}^{n} a_i}{n} = t \Rightarrow \sum a_i = tn$ if all $a_i < t$ then $\sum a_i < tn$ same for $> t$ and hence statement is clear.

4 Strong form of PHP

Let $q_1, q_2, ..., q_n \in \mathbb{Z}^+$. If $(q_1 + q_2 + ... + q_n - n + 1)$ objects are put into n boxes, then either the first box gets at least q_1 objects, or the second box gets at least q_2 objects or ..... or the n'th box gets at least q_n objects.

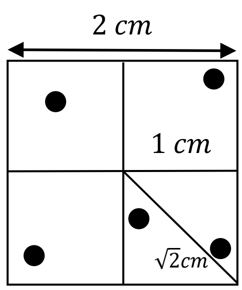
Proof: Put $q_i - 1$ objects in i'th box $\forall i$. Then # elements put in the box is $\sum(q_i - 1) =$

$\sum q_i - n$, now 1 element left, put it in any box and done.

5 Application of PHP

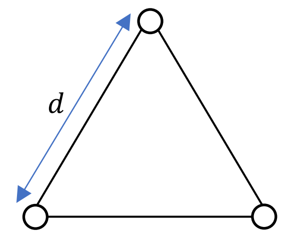
Example 1: Suppose you have a square of area 4 cm^2, if 5 points are put on the surface of it then at least 2 of them have distance $\leq \sqrt{2}$cm.

Sol.



Example 2: Suppose that every point in the real plane is colored either red or blue. Then for any distance $d > 0$, there are two points exactly distance d from one another that are the same color.

Sol. Consider any equilateral triangle whose side lengths are d. Put this triangle anywhere in the plane.



Now you have 3 points (your tree hole , remember the picture) to colour them with 2 colours (your pigeon).

Example 3: A chess master who has 11 weeks to prepare for a tournament decides to play at least one game every day but not to tire himself., he decides not to play more than 12 games during any calendar week. Show that there exists a succession of days during which the chess master will have played EXACTLY 21 games.

Sol. Suppose $a_i = \#$ the game played at the end of i'th day. The sequence $a_1, a_2, ..., a_{77}$ is strictly increasing $1 \leq a_1 < a_2 < ... < a_{77} \leq 11 \times 12 = 132$. Now add 21, then $22 \leq a_1{+}21 < .. < a_{77}{+}21 \leq 153$Now each of 154 numbers $a_1, .., a_{77}, a_1{+}21, ..., a_{77}{+}21$ is an integer between 1 and 153, then by PHP any two of $a_1, .., a_{77}, a_1 + 21, ..., a_{77} + 21$ are equal but $a_i \neq a_j$, also $a_i + 21 \neq a_j + 21 (i \neq j)$. Then we only can have option for following :- $\exists i \neq j; a_i = a_j + 21$(why $i \neq j$?) so, $a_i - a_j = 21$, hence we are done.

6 References:

- Introductory Combinatorics; Richard A. Brauldi; Fourth edition, PEARSON
- An Excursion in Mathematics; M R Modak, S A Katre, V V Acharya, V M Sholapurkar; Eleventh Edition - 2015, Bhaskaracharya Pratisthana, Pune