DEPARTMENT OF MATHEMATICS

NABADWIP VIDYASAGAR COLLEGE

PRITAM DAS

ROLL NO. 87

TOPIC:

Linear Differential Equations with constant coefficients

Introduction:

Linear differential equation occur so frequently in applications as to call for a somewhat detailed treatment which we propose to do.

The general from of a linear equation of n-th order is

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots \dots \dots . + P_{n-1} \frac{dy}{dx} + P_n y = Q$$

Where $P_1, P_2, \dots\dots\dots P_{n-1}, P_n$ and Q are constants or functions of x. In the present chapter , however, we shall assume that the coefficient $P_1, P_2, \dots\dots\dots P_{n-1}, P_n$ are constants. We shall consider simpler equations where Q is supposed to be 0.

Equations of First Order

<u>The Simple Case</u>

In let us take n=1 and Q=0 we then obtain

$$\frac{dy}{dx} + p_1 y = 0$$

Separating the variables we easily get $y = Ae^{-p_1 x}$, where A is an arbitrary constant.

Equation with right-hand member zero

Let the equation be

$$\frac{d^2 y}{dx^2} + p_1 \frac{dy}{dx} + p_2 y = 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . . (1)$$

Let us take $y = ce^{mx}$ (≠) be the non-trial solution.

Then if we put $y=ce^{mx}$ in the left side of (1), it must satisfy the equation, i.e. ,we must have

$$C(m^2+p_1+p_2)e^{mx} =0$$

Or, since $ce^{mx} \neq 0$, $m^2+p_1+p_2=o$..........................(2)

The equation (2) is called the Auxiliary equation (1)

Then , $y=ce^{m_1}x$ and $y = ce^{m_2}x$ are obvious solution of (2).

Also, it can be easily verified by direction substitution that $y=c_1e^{m_1 x}$, $y=c_2e^{m_1}$ and satisfy the equation (1) and such are solution of (1).

We shall now consider the nature the nature the general solution of (1) according as the roots of the auxiliary equation (2) are real distinct real and equal and imaginary.

Method of Variation of Parameters

We have already introduced the method of Variation Parameters in solving linear equation of first order and first degree. Let us extend this case to a second order linear differential equation of higher order.

To discuss the solution of an n^{th} order linear differential equation of the from

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots \ldots \ldots . + P_{n-1}\frac{dy}{dx} + P_n = R(x)$$

by the method of variation of parameters .

let, $y_1(x)$, $y_2(x)$, ..., $y_n(x)$ be a set of n linearly independent solution to the associated homogeneous differential equation –that is the equation (1) itself with R.H.S. R(x) = 0 . The C.F. is

$$c_1y_1 + c_2y_2 + \ldots\ldots\ldots + c_ny_n$$

where $c_1, c_2, c_3, \ldots\ldots\ldots, c_n$ are arbitrary constants . Here $p_1(x)$, $p_2(x)$,.......,$p_n(x)$ need not constant only

A particular solution y_p to (1) is

$$Y_p = v_1y_1 + v_2y_2 + \ldots\ldots\ldots + v_ny_n$$

Where $v_1, v_2, \ldots\ldots\ldots, v_n$ are unknown function to be determine.

As before these can be found by solving

$$V_1^{/}y_1 + v_2^{/}y_2 + \ldots\ldots\ldots + v_n^{/}y_n = 0$$

$$V_1^{/}y^{/}_1 + v_2^{/}y^{/}_2 + \ldots\ldots\ldots + v_n^{/}y^{/}_n = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$V_1^{/}y_1^{n-2} + v_2^{/}y_2^{n-2} + \ldots\ldots\ldots + v_n^{/}y_n^{n-2} = 0$$

$$V_1^{/}y_1^{n-1} + v_2^{/}y_2^{n-1} + \ldots\ldots\ldots + v_n^{/}y_n^{n-1} = 0$$

Since from these equation we obtain $v_1^{/}, v^{/}_2, \ldots\ldots\ldots, v_n^{/}$ and by integration we have the values $v_1, v_2, \ldots\ldots\ldots v_n$.

Thus the general solution is given by

$$Y = c_1y_1 + c_2y_2 + \ldots\ldots\ldots + c_ny_n + y_p$$

Example: Solve by the method of variation of parameter:

$$(D^2 + a^2)\, y = \tan ax$$

Solution: To find C.F.

Auxiliary equation is $(m^2+a^2)=0$ which gives $m=\pm ia$,

The C.F. is $C_1 cosax+C_2 sinax$

To find P.I.

call y_1= cosax, y_2= sinax and it has observed that y_1 ,y_2, are independent solutions. Also the particular solution will be of the form

$$Y_p=v_1(x)cosax+v_2(x)sinax$$

$$Dy_p=-av_1(x)sinax+av_2(x)cosax+v_1'(x)cosax+v_2'(x)sinax$$

Next proceed as before where $v_1'(x)$, $v_2'(x)$ can be found from

$$V_1'(x)cosax+v_2'(x)sinax=0$$

$$(D^2+a^2)y_p=-av_1'(x)sinax+av_2'(x)cosax$$

$$tanax=-av_1'(x)sinax+av_2'(x)cosax$$

The Wronskians of y_1 and y_2 is

$$W(y_1,y_2)=\begin{vmatrix} cosax & sinax \\ -asinax & acosax \end{vmatrix}=a\neq 0$$

$$V_1(x)=\int(-sinaxtanax/a)dx=-(1/a^2)\log|secax + tanax|+(1/a^2)sinax$$

$$V_2(x)= \int(cosaxtanax/a)dx=-(1/a^2)cosax$$

$$Y_p=(-1/a^2)cosax\log|secax + tanax|$$

Complete solution is $y= C_1cosax+C_2sinax-(1/a^2)cosax\log|secax + tanax|$

References

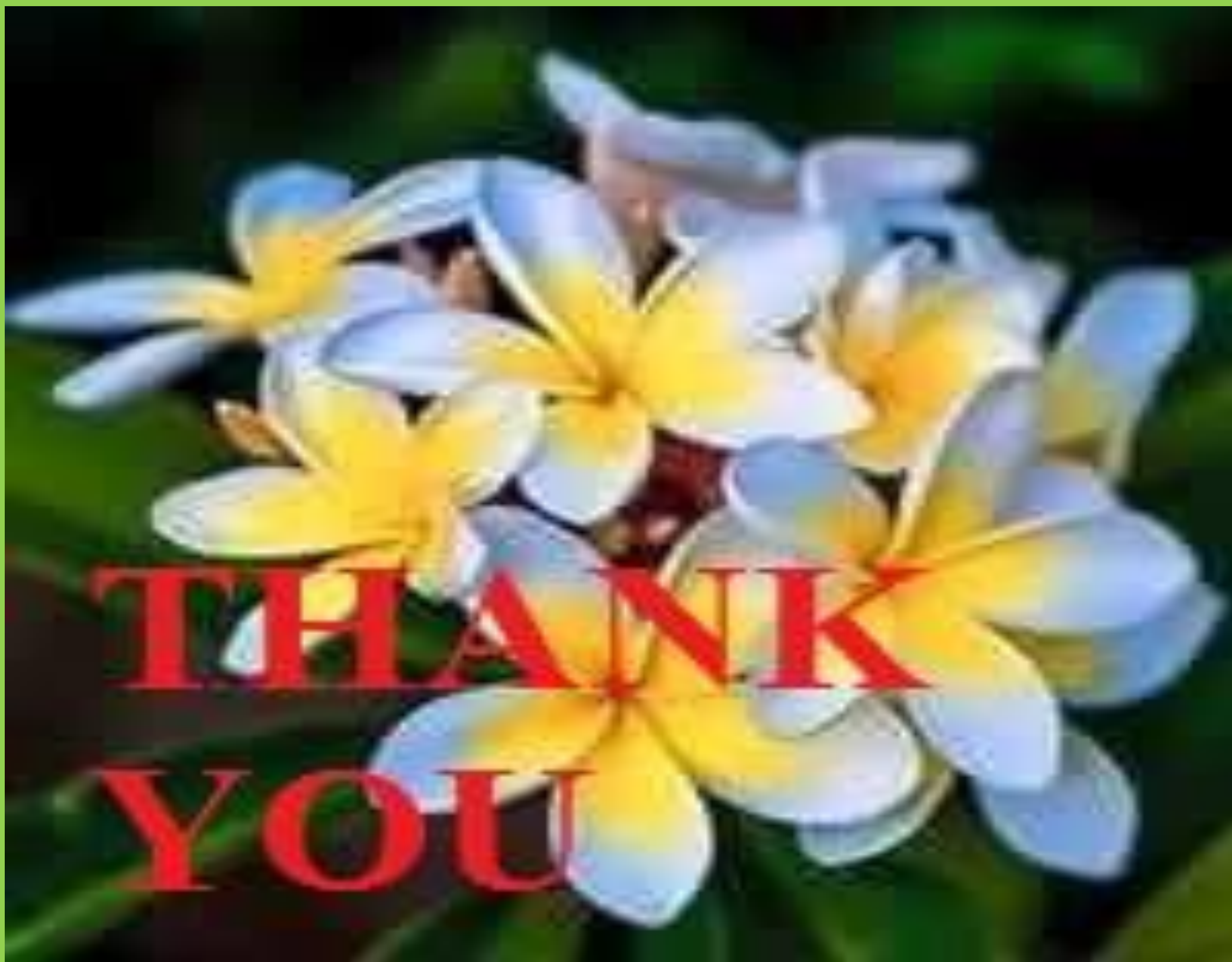
1 : M.D.Raisihania: Ordinary and Partial Differential Equations

2 : R.K.Ghosh and K.C. Maity: An Introduction to Differential Equations

3 : Chakraborty and Ghosh : Differential Equations.

4 : Murray : Differential Equations.

5 : Piaggio : Differential Equations.


THANK
YOU