# Continuity of Functions of a Single Real Variable

## Departmental Seminar on Mathematics and its Applications

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In defining the symbol $\lim_{x \to c} f(x)$ , we did not make use of the value of f at x=c (in fact ,such a value may not be defined ). Even if f is defined at x=c , the value of f(c) need not always equal to the limiting value l . When we do have f(c) =l , i. e. ,when $\lim_{x \to c} f(x)$ =f(c) ,we say that f is continuous at x=c . In the present chapter we shall study some deeper properties of continuous function defined over a certain interval I (Open or closed)

# Continuity at a point:

Let $f$ be defined on an interval $I \subseteq R$
Suppose $c$ is an interval point of $I$ . The continuity of $f$ at $x = c$ may be defined in the following way :
Definition: $f$ is said to be continuous at $x = c$ if for any arbitrary positive number $\epsilon$ (i.e,$\epsilon > 0$) no mater how small , $\exists$ a positive number $\delta (\delta > 0)$ such that $|f(x) - f(c)| < \epsilon$ for all points $x \in |x - c| < \delta$
It means that if $f$ is continuous at $c$ then $\lim_{x \longrightarrow c} f(x)$ exists and is equal to $f(c)$ .

# Continuity at a point:

This means that f is continuous at $x = c$ if the functions values $f(x)$ are close to $f(c)$ when $x$ is close to $c$ ,thus for continuity of $f$ at $x = c$,$f$ must be defined at $x = c$ and also in some neighbourhood around $c$, otherwise. We can not find $\underset{x \to c}{lim} \ f(x)$ which must exist and equals to $f(c)$ . Hence for continuity of $f$ at a point $x = c$, we must have

1. $f$ is defined at $c$ i.e, $f(c)$ exists.
2. $\lim_{x \to c} f(x)$ exists i.e, $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$
3. $f(c)$ and $\underset{x \to c}{lim} \ f(x)$must be equal .

# Definition

If any of these condition fail, then $f$ is not continuous at $c$ or $f$ is discontinuous at $x = c$.
Sometimes a functions $f$ may be continuous on one side of $c$ .We call such a continuity as one side continuity. More explicitly,

## Definition

(1) $f$ is said to be right continuous at point $x = a$ ,if $\lim\limits_{x \to a^{+0}} f(x)=f(a)$,i.e,given any $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in (a, a + \delta)$ we have $|f(x) - f(a)| < \epsilon$.Here f must be defined in some right neighbourhood of $a$ . (2) $f$ is said to be left continuous at a point $x = b$ if $\lim\limits_{x \to b^{-0}} f(x) = f(b)$ ,i.e, given any $\epsilon > 0, \exists \delta > 0$ such that for all $x \in (b\text{-}\delta, b)$ we have $|f(x) - f(b)| < \epsilon$ Here $f$ must be defined in some left neighbourhood of $b$.

# Continuity in an interval :

Continuity of $f$ in an open interval $(a,b)$ : $f$ is continuous at every point of $c$, where $a < c < b$ ,
Continuity of $f$ in a closed interval $[a,b]$:

1. If c is an interior point of $[a,b]$, then $f$ is continuous at $x = c$.
2. At the left end point $a$ , $f$ is right continuous.
3. At the right end point $b$, $f$ is left continuous.

The definition of continuity at $a$ point can also be defined in the language of sequence.

# Example:

## Example

We are to show that $f(x) =$sin$x$ is continuous for all values of $x$ . Here also the domain of $f(x) =$sin$x$ is $R$, the set of all real numbers . Take any value a of $x$ , where $a \in R$.sin$x$ will be continuous for $x = a$ if for any $\epsilon > 0$ , we can find a value of $\delta$>0 such that |sin$x$−sin$a$| $< \epsilon$ whenever$|x - a| < \delta$.Now

$$|\sin x - \sin a| = |2\sin(x-a)/2\cos(x+a)/2| = 2|\sin(x-a)/2||cos(x+a)/$$

As $|\cos(x+a)/2| \leq 1$ for every value of $x \in R$ and $|\sin(x-a)/2| < |(x-a)/2|$, for $0 < |(x-a)/2| < \pi/2[\therefore |\sin x| < |x| < |\tan x|$ *for* $0 < x < \pi/2]$

$$|\sin x - \sin a| = 2|\sin(x-a)/2||\cos(x+a)/2| < 2|(x-a)/2| = |x-a|$$

$|\sin x - \sin a| < \epsilon$ for all x for which $|x - a| < \delta$,where $\delta = \epsilon$. i.e, $\sin x$ is continuous at any arbitrary point $a \in R \Rightarrow$sinx is continuous for all $x \in \mathbb{R}$.

# References


- Ghosh and Maity : An Introduction to Analysis-Differential Calculus.
- Shanti Narayan : Differential Calculus.
- Das and Mukherjee : Differential Calculus.
- S.C.Malik and S. Arora: Mathematical Analysis
- S.K.Mapa: Introduction to Real Analysis

# THANK YOU