U.G. 6th Semester Examination - 2023

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-14

(Ring Theory and Linear Algebra)

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

The symbols and notations have their usual meanings.

1. Answer any **ten** questions: $2\times10=20$

a) Is $f:(\mathbb{Z},+,.)\to(2\mathbb{Z},+,.)$ defined by $f(n)=2n,\ n\in\mathbb{Z}$ a ring homomorphism? Justify your answer.

b) Let R be a ring with unity 1 and $\phi:R\to R'$ be a ring homomorphism. If $\phi(1)=0$, prove that kernel of $\phi=R$.

c) Show that the mapping $f:\mathbb{Z}[\sqrt{2}]\to\mathbb{Z}[\sqrt{3}]$ defined by

$$f\left(a+b\sqrt{2}\right)=a+b\sqrt{3},\ a+b\sqrt{2}\in\mathbb{Z}\left[\sqrt{2}\right]$$

is a group homomorphism but not a ring homomorphism.

d) Show that $\frac{\mathbb{Z}}{2\mathbb{Z}}\simeq\frac{5\mathbb{Z}}{10\mathbb{Z}}$.

e) Show that quotients field of $2\mathbb{Z}$ is same as the field of quotients of $\mathbb{Z}$.

f) If F is a field, show that $F[x]$ is an integral domain but not a field.

g) Is $\frac{\mathbb{Z}[x]}{\langle x\rangle}$ a field? Justify your answer.

h) Find the characteristic polynomial of the linear operator $D:V\rightarrow V$ defined by $D(f)=\frac{df}{dt}$, where V is the space of functions with basis $S=\{\sin t, \cos t\}$.

i) If λ is an eigen value of an invertible linear operator T, show that λ^{-1} is an eigen value of T^{-1}.

j) Find the minimal polynomial of the matrix $A=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$.

k) Show that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

l) If $T: V \to V$ be a linear mapping, show that kernel of T is invariant under T.

m) Let S_1, S_2 are subsets of a vector space V and $S_1 \subseteq S_2$. Prove that $S_2^{\perp} \subseteq S_1^{\perp}$.

n) Show that an orthogonal set of non-null vectors in a Euclidean space V is linearly independent.

o) Prove that for all α, β in a Euclidean space V, $\langle \alpha, \beta \rangle = 0$ if and only if $\| \alpha + \beta \| = \| \alpha - \beta \|$.

2. Answer any **four** questions: 5×4=20

a) Let R and R' be two rings and $\phi : R \to R'$ be a ring homomorphism. Define Ker ϕ and show that Ker ϕ is an ideal of R. 1+4

b) State and prove Fundamental theorem of ring homomorphism.

c) Show that $\mathbb{Z}[x]$ is an integral domain but not a principal ideal domain.

d) Prove that for prime number p, $f(x) = x^{p-1} + x^{p-2} + ... + x + 1$ is irreducible in $\mathbb{Q}[x]$.

e) Let $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a basis of a vector space V over a field F and $\phi_1, \phi_2, ..., \phi_n \in V^*$ be the linear functionals defined by

$$\phi_i(\alpha_j) = \begin{cases} 1, & if \quad i = j \\ 0, & if \quad i \neq j. \end{cases}$$

Show that $\{\phi_1, \phi_2, ..., \phi_n\}$ is a basis of V^*.

f) Find all eigen values and a basis of each eigen space of the operator $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (2x + y,\ y - z,\ 2y + 4z)$.

g) Let $T: V \to U$ be linear and $T': U^* \to V^*$ be its transpose. Show that kernel of T' is the annihilator of the image of T.

3. Answer any **two** questions: 10×2=20

a) i) Let F be a field $f(x)$ be a polynomial in $F[x]$ of degree 2 or 3. Prove that $f(x)$ is irreducible if and only if it has a zero in F.

ii) Show that the polynomial $x^2 + x + 1$ is irreducible in $\mathbb{Z}_2[x]$.

iii) Show that the quotient ring $\dfrac{\mathbb{Z}_2[x]}{\langle x^2 + x + 1 \rangle}$ is a field of 4 elements. 4+2+4

b) i) Let V be the vector space of polynomials over $\mathbb{R}$ of degree ≤ 1. Let ϕ_1 and ϕ_2 be linear functionals defined by

$$\phi_1[f(x)] = \int_0^1 f(x)\,dx \quad \text{and} \quad \phi_2[f(x)] = \int_0^2 f(x)\,dx.$$

Find the basis $\{f_1, f_2\}$ of V which is dual to $\{\phi_1, \phi_2\}$.

ii) Let W be the subspace of $\mathbb{R}^4$ spanned by $v_1 = (1, 2, -3, 4)$ and $v_2 = (0, 1, 4, -1)$. Find a basis of the annihilator of W.

5+5

c) i) Find all ring homomorphisms from $\mathbb{Z}_{20} \to \mathbb{Z}_{30}$.

ii) Find the quotient field of the integral domain $\mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} : a, b \in \mathbb{Z}\}$.

iii) Let $v_1, v_2, \ldots, v_n$ be non-zero eigen vectors of an operator $T : V \to V$ belonging to distinct eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$. Prove that $v_1, v_2, \ldots, v_n$ are linearly independent.

3+3+4

d) i) Determine all possible Jordan canonical forms for a linear operator $T : V \to V$ whose characteristic polynomial is $(t-2)^3 (t-5)^2$.

ii) Let V be a vector space of polynomials $f(t)$ of degree less than 4 and real coefficients. Define inner product on V as $\langle f, g \rangle = \int_{-1}^{1} f(t) g(t)\, dt$. Apply the Gram-Schmidt orthogonalization process to $\{1, t, t^2, t^3\}$ to find an orthogonal basis $\{f_0, f_1, f_2, f_3\}$ of V. 5+5
